JEE Advanced 2021 Question Paper with Solution

Joint Entrance Examination – Advanced

DATE : 03/10/2021

Time : 3 hrs. **Answers & Solutions** Max. Marks: 180

for

JEE (Advanced)-2021

PAPER - 1

PART-I : PHYSICS

SECTION - 1

- This section contains **FOUR (04)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks	:	+3	If ONLY the correct option is chosen;
Zero Marks	:	0	If none of the options is chosen (i.e. the question is unanswered);
Negative Marks	:	–1	In all other cases.

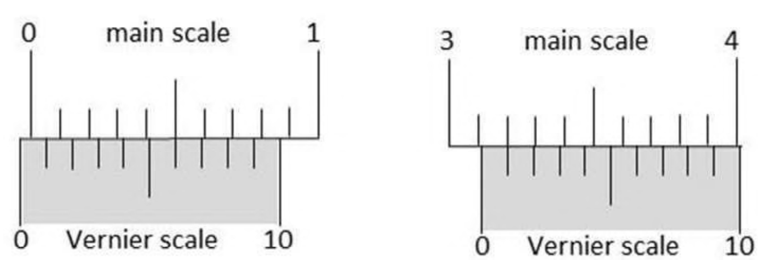
1. The smallest division on the main scale of a Vernier calipers is 0.1 cm. Ten divisions of the Vernier scale correspond to nine divisions of the main scale. The figure below on the left shows the reading of this calipers with no gap between its two jaws. The figure on the right shows the reading with a solid sphere held between the jaws. The correct diameter of the sphere is

(A) 3.07 cm

(B) 3.11 cm

(C) 3.15 cm

(D) 3.17 cm



Answer (C)

Sol. Least count of Vernier calipers = 0.01 cm

Error in scale = 4 LC

= 0.04 cm

Reading = 3.1 cm + 1 L.C

= 3.1 cm + 0.01

= 3.11 cm

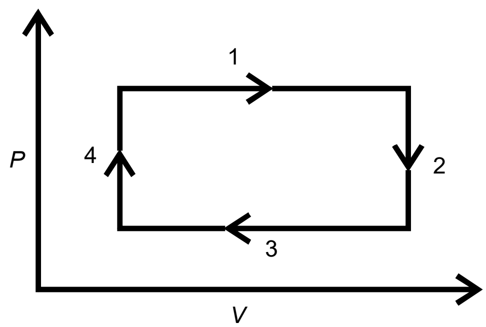
So correct diameter of the sphere

= (3.11 + 0.04) cm

= 3.15 cm

So, option (C)

2. An ideal gas undergoes a four step cycle as shown in the $P - V$ diagram below. During this cycle, heat is absorbed by the gas in



(A) steps 1 and 2

(B) steps 1 and 3

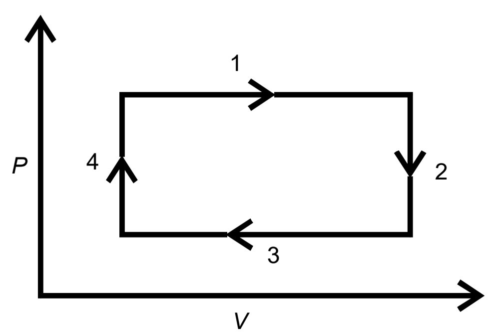
(C) steps 1 and 4

(D) steps 2 and 4

Answer (C)

Sol. Given $P - V$ diagram

For process (1)



$\Delta Q_1 = nC_P \Delta T$

As P = constant and V increases

so T will increase

So $\Delta Q_1 > 0$

For process (2)

$\Delta Q_2 = nC_V \Delta T$

V = constant, $P\downarrow$, So $T\downarrow$

For process (3), $\Delta Q_3 = nC_P \Delta T < 0$

For process (4), $\Delta Q_4 = nC_P \Delta T$

As $\Delta T > 0$

So $\Delta Q_4 > 0$

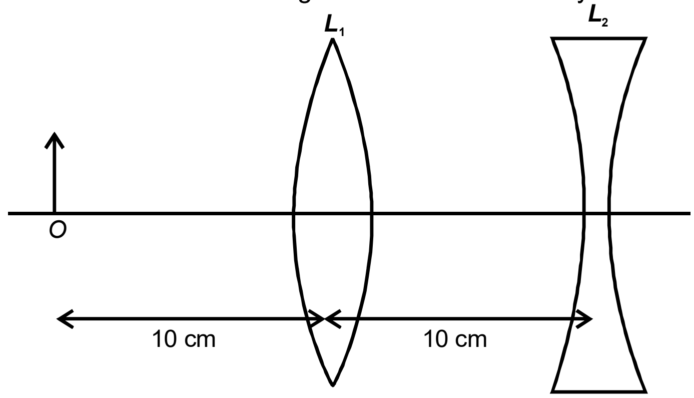
3. An extended object is placed at point O, 10 cm in front of a convex lens L_1 and a concave lens L_2 is placed 10 cm behind it, as shown in the figure. The radii of curvature of all the curved surfaces in both the lenses are 20 cm. The refractive index of both the lenses is 1.5. The total magnification of this lens system is

(A) 0.4

(B) 0.8

(C) 1.3

(D) 1.6



Answer (B)

Sol. $\dfrac{1}{v_1} - \dfrac{1}{-10} = \dfrac{1}{20}$

$\Rightarrow \dfrac{1}{v_1} = \dfrac{1}{20} - \dfrac{1}{10} = \dfrac{-1}{20}$

$\Rightarrow v_1 = -20\,\text{cm}$

$m_1 = \dfrac{v}{u_1} = \dfrac{-20}{-10} = 2$

again $\dfrac{1}{v_2} - \dfrac{1}{-30} = \dfrac{1}{-20}$

$\Rightarrow \dfrac{1}{v_2} = -\dfrac{1}{30} - \dfrac{1}{20} = -\dfrac{5}{60} = -\dfrac{1}{12}$

$m_2 = -\dfrac{12}{-30} = \dfrac{2}{5}$

$m = m_1 \times m_2 = 2 \times \dfrac{2}{5}$ = 0.8

$\dfrac{1}{f_1} = \left(\dfrac{3}{2} - 1\right)\left(\dfrac{2}{20}\right)$

$= \dfrac{1}{20}$

4. A heavy nucleus Q of half-life 20 minutes undergoes alpha-decay with probability of 60% and beta-decay with probability of 40%. Initially, the number of Q nuclei is 1000. The number of alpha-decays of Q in the first one hour is

(A) 50

(B) 75

(C) 350

(D) 525

Answer (D)

Sol: $t_{1/2}$ = 20 min

In 60 min, no. of half-life = 3

$$\Rightarrow \quad N_A = \left[1000 - \frac{1000}{2^3}\right] \times 0.6$$

$$= \quad 1000 \times \frac{7}{8} \times 0.6$$

$$= \quad 525$$

SECTION - 2

- This section contains **THREE (03)** question stems.
- There are **TWO (02)** questions corresponding to each question stem.
- The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical value corresponding to the answer in the designated place using the mouse and the on-screen virtual numeric keypad.
- If the numerical value has more than two decimal places, **truncate/round-off** the value to **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks	:	+2	If ONLY the correct numerical value is entered at the designated place;
Zero Marks	:	0	In all other cases.

Question Stem for Question Nos. 5 and 6

Question Stem

A projectile is thrown from a point *O* on the ground at an angle 45° from the vertical and with a speed $5\sqrt{2}\,\text{m/s}$. The projectile at the highest point of its trajectory splits into two equal parts. One part falls vertically down to the ground, 0.5 s after the splitting. The other part, *t* seconds after the splitting, falls to the ground at a distance *x* meters from the point *O*. The acceleration due to gravity *g* = 10 m/s^2.

5. The value of *t* is ________.

Answer (00.50)

Sol. $H = \dfrac{u^2 \sin^2\theta}{2g}$

$$= \frac{50}{2\times 10} \times \frac{1}{2} = \frac{5}{4}$$

$$t = \sqrt{\frac{2H}{g}} = \sqrt{\frac{2\times 5}{4\times 10}}$$

$$t = \frac{1}{2}\,\text{s} = 0.5\,\text{sec}$$

Ans. 00.50

6. The value of x is ________.

Answer (07.50)

Sol. $X = \frac{3R}{2}$ as $X_{cm} = R$

$$R = \frac{u^2 \sin^2 \theta}{g} = \frac{50}{10} = 5$$

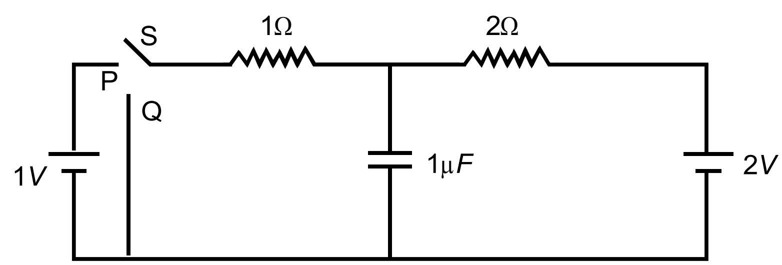
$$\Rightarrow \quad X = \frac{3R}{2} = \frac{15}{2} = 7.5 \text{ m}$$

Ans.: 07.50

Question Stem for Question Nos. 7 and 8

Question Stem

In the circuit shown below, the switch S is connected to position P for a long time so that the charge on the capacitor becomes q_1 μC. Then S is switched to position Q. After a long time, the charge on the capacitor is q_2 μC.



7. The magnitude of q_1 is_____.

Answer (01.33)

Sol. With switch S at position P after long time potential difference across capacitor branch

$$= \frac{2/2 + 1/1}{1/2 + 1/1} = \frac{2 \times 2}{3} = \frac{4}{3} V$$

$$\Rightarrow \quad \text{Charge on capacitor } q_1 \mu C = \frac{4}{3} \mu C$$

$$\Rightarrow \quad q_1 = \frac{4}{3} = 1.33$$

8. The magnitude of q_2 is_____.

Answer (00.67)

Sol. With switch S at position Q after long time potential difference across capacitor

= potential difference across resistance of 1 ohm.

$$= \frac{2}{3} V$$

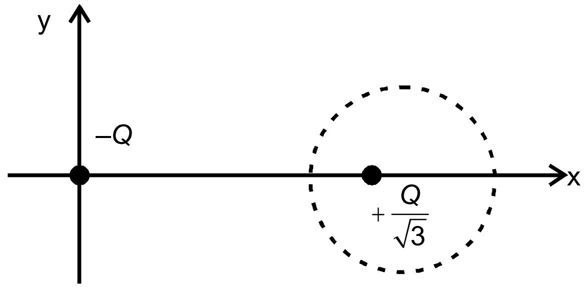
$\Rightarrow$ charge on capacitor $q_2 \mu C = \frac{2}{3} \mu C$

$\Rightarrow$ q_2 = 0.67

Question Stem for Question Nos. 9 and 10

Question Stem

Two point charges –Q and $+Q/\sqrt{3}$ are placed in the *xy*-plane at the origin (0, 0) and a point (2, 0), respectively, as shown in the figure. This results in an equipotential circle of radius R and potential V = 0 in the *xy*-plane with its center at (b, 0). All lengths are measured in meters.



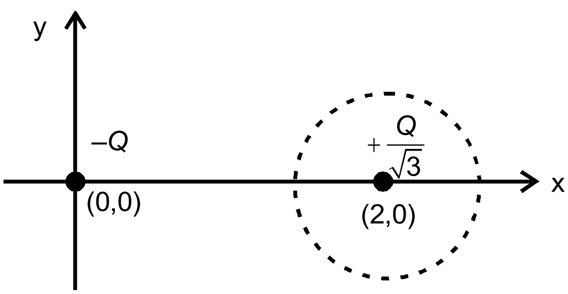
9. The value of R is ___ meter

Answer (R = 01.73)

10. The value of b is ___ meter

Answer (b = 03.00)

Solution for Q. Nos. 9 & 10



$$V(x,y) = \frac{1}{4\pi\varepsilon_0}\left(-\frac{Q}{\sqrt{x^2+y^2}} + \frac{Q}{\sqrt{3}\sqrt{(x-2)^2+y^2}}\right)$$

$$\Rightarrow \quad 3(x-2)^2 + 3y^2 = x^2 + y^2$$

$$\Rightarrow \quad (x-3)^2 + y^2 = (\sqrt{3})^2$$

SECTION - 3

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONE OR MORE THAN ONE** of these four option(s) is (are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).

· Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +4 If only (all) the correct option(s) is(are) chosen;

Partial Marks : +3 If all the four options are correct but ONLY three options are chosen;

Partial Marks : +2 If three or more options are correct but ONLY two options are chosen, both of which are correct;

Partial Marks : +1 If two or more options are correct but ONLY one option is chosen and it is a correct option;

Zero Marks : 0 If unanswered;

Negative Marks : – 2 In all other cases.

· For example, in a question, if (A), (B) and (D) are the ONLY three options corresponding to correct answers, then

Choosing ONLY (A), (B) and (D) will get +4 marks;

Choosing ONLY (A) and (B) will get +2 marks;

Choosing ONLY (A) and (D) will get +2 marks;

Choosing ONLY (B) and (D) will get +2 marks;

Choosing ONLY (A) will get +1 mark;

Choosing ONLY (B) will get +1 mark;

Choosing ONLY (D) will get +1 mark;

Choosing no option(s) (i.e. the question is unanswered) will get 0 marks and choosing any other option(s) will get “2 marks.

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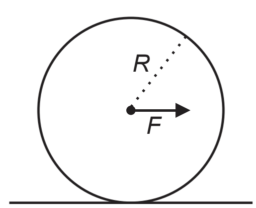
11. A horizontal force F is applied at the center of mass of a cylindrical object of mass m and radius R, perpendicular to its axis as shown in the figure. The coefficient of friction between the object and the ground is μ. The center of mass of the object has an acceleration a. The acceleration due to gravity is g. Given that the object rolls without slipping, which of the following statement(s) is(are) correct?



(A) For the same F, the value of a does not depend on whether the cylinder is solid or hollow

(B) For a solid cylinder, the maximum possible value of a is $2\mu g$

(C) The magnitude of the frictional force on the object due to the ground is always μmg

(D) For a thin-walled hollow cylinder, $a = \frac{F}{2m}$

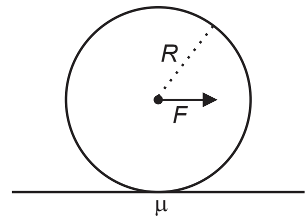
Answer (B, D)

Sol. For solid cylinder,

$$F \times R = \frac{3}{2} mR^2 \times \left(\frac{a}{R}\right)$$

$$\Rightarrow \quad a = \frac{2F}{3m}$$

For hollow cylinder,



$$F \times R = (2mR^2) \times \frac{a}{R}$$

$$\Rightarrow \quad a = \frac{F}{2m}$$

For solid cylinder

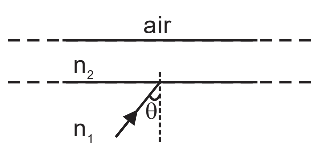
$$f = F - m \times \frac{2F}{3m} = \frac{F}{3} \le \mu mg$$

$$\Rightarrow \quad F \le 3\mu mg$$

$$\therefore \quad a \le \frac{2}{3m} \times (3\,\mu mg)$$

$$\Rightarrow \quad a_{max} = 2\mu g$$

12. A wide slab consisting of two media of refractive indices n_1 and n_2 is placed in air as shown in the figure. A ray of light is incident from medium n_1 to n_2 at an angle θ, where $\sin\theta$ is slightly larger than $\frac{1}{n_1}$. Take refractive index of air as 1. Which of the following statement(s) is(are) correct?



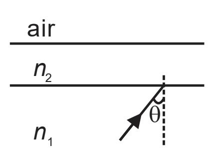
(A) The light ray enters air if $n_2 = n_1$

(B) The light ray is finally reflected back into the medium of refractive index n_1 if $n_2 < n_1$

(C) The light ray is finally reflected back into the medium of refractive index n_1 if $n_2 > n_1$

(D) The light ray is reflected back into the medium of refractive index n_1 if $n_2 = 1$

Answer (B, C, D)



Sol. $\sin\theta > \frac{1}{n_1}$ ... (i)

and, $n_1\sin\theta = 1 \times \sin r$ ... (ii)

$\Rightarrow$ $\sin r > 1$

$\Rightarrow$ refraction into air is not possible.

13. A particle of mass M = 0.2 kg is initially at rest in the xy-plane at a point (x = $-l$, y = $-h$), where l = 10 m and h = 1 m. The particle is accelerated at time t = 0 with a constant acceleration a = 10 m/s^2 along the positive x-direction. Its angular momentum and torque with respect to the origin, in SI units, are represented by $\vec{L}$ and $\vec{\tau}$, respectively. $\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vectors along the positive x, y and z-directions respectively. If $\hat{k} = \hat{i} \times \hat{j}$ then which of the following statement(s) is(are) correct?

(A) The particle arrives at the point (x = l, y = $-h$) at time t = 2s

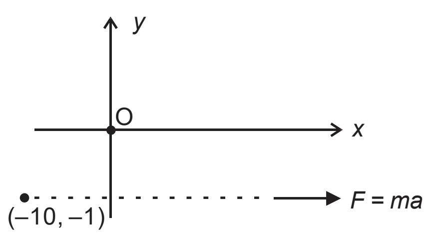
(B) $\vec{\tau} = 2\hat{k}$ when the particle passes through the point (x = l, y = $-h$)

(C) $\vec{L} = 4\hat{k}$ when the particle passes through the point (x = l, y = $-h$)

(D) $\vec{\tau} = \hat{k}$ when the particle passes through the point (x = 0, y = $-h$)

Answer (A, B, C)

Sol.



$$t = \sqrt{\frac{2 \times (20)}{10}} = 2\text{s}$$

$$\vec{\tau} = (0.2 \times 10 \times 1)\,\hat{k} = 2\hat{k}$$

$$\vec{L} = [0.2 \times (10 \times 2) \times 1]\,\hat{k} = 4\hat{k}$$

14. Which of the following statement(s) is(are) correct about the spectrum of hydrogen atom?

(A) The ratio of the longest wavelength to the shortest wavelength in Balmer series is $\frac{9}{5}$

(B) There is an overlap between the wavelength ranges of Balmer and Paschen series

(C) The wavelengths of Lyman series are given by $\left(1 + \frac{1}{m^2}\right)\lambda_0$, where λ_0 is the shortest wavelength of Lyman series and m is an integer

(D) The wavelength ranges of Lyman and Balmer series do not overlap

Answer (A, D)

Sol. For Balmer series :

$$\frac{1}{\lambda} = R\left(\frac{1}{2^2} - \frac{1}{n^2}\right) \qquad n = 3, 4, 5.....$$

$$\frac{1}{\lambda_{max}} = R\left(\frac{1}{4} - \frac{1}{9}\right)$$

$$\frac{1}{\lambda_{min}} = R\left(\frac{1}{4}\right)$$

$$\Rightarrow \quad \frac{\lambda_{max}}{\lambda_{min}} = \frac{9}{5}$$

For Lyman series

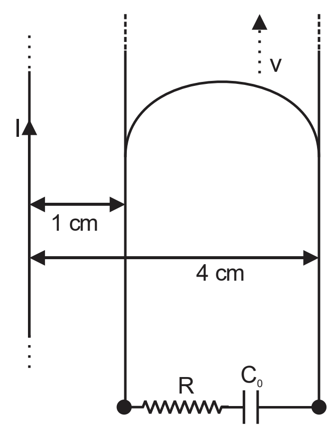
$$\frac{1}{\lambda} = R\left(1 - \frac{1}{n^2}\right) \qquad n = 2,3,4......$$

$$\frac{1}{\lambda_{min}} = R$$

$$\Rightarrow \quad \lambda = \frac{\lambda_0 n^2}{n^2 - 1}$$

15. A long straight wire carries a current, I = 2 ampere. A semi-circular conducting rod is placed beside it on two conducting parallel rails of negligible resistance. Both the rails are parallel to the wire. The wire, the rod and the rails lie in the same horizontal plane, as shown in the figure. Two ends of the semi-circular rod are at distances 1 cm and 4 cm from the wire. At time t = 0, the rod starts moving on the rails with a speed v = 3.0 m/s (see the figure).

A resistor R = 1.4 Ω and a capacitor C_0 = 5.0 μF are connected in series between the rails. At time t = 0, C_0 is uncharged. Which of the following statement(s) is(are) correct? [$\mu_0 = 4\pi \times 10^{-7}$ SI units. Take ln 2 = 0.7]



(A) Maximum current through R is 1.2×10^{-6} ampere

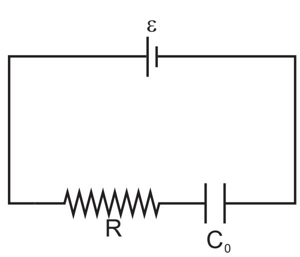
(B) Maximum current through R is 3.8×10^{-6} ampere

(C) Maximum charge on capacitor C_0 is 8.4×10^{-12} coulomb

(D) Maximum charge on capacitor C_0 is 2.4×10^{-12} coulomb

Answer (A, C)

Sol. Equivalent circuit of the given arrangement is :



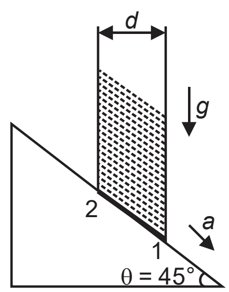
Where $\varepsilon = \frac{\mu_0 Iv}{2\pi} \ln \frac{b}{a}$

$= 1.68 \times 10^{-6}$ V

At t = 0, $i_{max} = \frac{\varepsilon}{R} = \frac{1.68 \times 10^{-6}}{1.4} = 1.2 \times 10^{-6}$ **A**

At $t = \infty$, $q_{max} = C_0\varepsilon = 8.4 \times 10^{-12}$ C

16. A cylindrical tube, with its base as shown in the figure, is filled with water. It is moving down with a constant acceleration a along a fixed inclined plane with angle θ = 45º. P_1 and P_2 are pressures at points 1 and 2, respectively, located at the base of the tube. Let $\beta = \frac{(P_1 - P_2)}{(\rho g d)}$, where ρ is density of water, d is the inner diameter of the tube and g is the acceleration due to gravity. Which of the following statement(s) is(are) correct?



(A) $\beta = 0$ **when** $a = \frac{g}{\sqrt{2}}$

(B) $\beta > 0$ **when** $a = \frac{g}{\sqrt{2}}$

(C) $\beta = \frac{\sqrt{2}-1}{\sqrt{2}}$ **when** $a = \frac{g}{2}$

(D) $\beta = \frac{1}{\sqrt{2}}$ **when** $a = \frac{g}{2}$

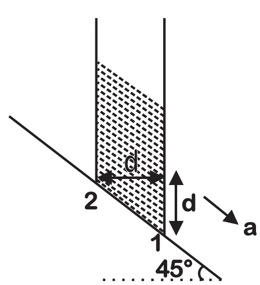
Answer (A, C)

Sol. $P_1 = P_2 - \rho a\cos 45° d + \rho(g - a\sin 45°)d$

$\Rightarrow \frac{P_1 - P_2}{\rho g d} = 1 - \frac{\sqrt{2}a}{g}$

$\Rightarrow \beta = 0$ **for** $a = \frac{g}{\sqrt{2}}$

$\beta = \frac{\sqrt{2}-1}{\sqrt{2}}$ **for** $a = \frac{g}{2}$



SECTION 4

- This section contains **THREE (03)** questions.
- The answer to each question is a **NON-NEGATIVE INTEGER.**
- For each question, enter the correct integer corresponding to the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +4 If ONLY the correct integer is entered

Zero Marks : 0 In all other cases.

17. An α-particle (mass 4 amu) and a singly charged sulfur ion (mass 32 amu) are initially at rest. They are accelerated through a potential V and then allowed to pass into a region of uniform magnetic field which is normal to the velocities of the particles. Within this region, the α-particle and the sulfur ion move in circular orbits of radii r_α and r_s, respectively. The ratio $\left(\frac{r_s}{r_\alpha}\right)$ is ________.

Answer (4)

Sol. $r = \frac{mv_0}{qB}$

$$\frac{1}{2}mv_0^2 = qV$$

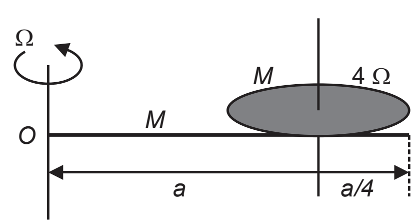
$$r = \frac{\sqrt{2mqV}}{qB}$$

$$r = \frac{1}{B}\sqrt{\frac{2mV}{q}}$$

$$\frac{r_s}{r_\alpha} = \sqrt{\frac{m_s}{q_s}} \times \sqrt{\frac{q_\alpha}{m_\alpha}} = \sqrt{2} \times \sqrt{8}$$

$$\frac{r_s}{r_\alpha} = 4$$

18. A thin rod of mass M and length a is free to rotate in horizontal plane about a fixed vertical axis passing through point O. A thin circular disc of mass M and of radius $\frac{a}{4}$ is pivoted on this rod with its center at a distance $\frac{a}{4}$ from the free end so that it can rotate freely about its vertical axis, as shown in the figure. Assume that both the rod and the disc have uniform density and they remain horizontal during the motion. An outside stationary observer finds the rod rotating with an angular velocity Ω and the disc rotating about its vertical axis with angular velocity 4Ω. The total angular momentum of the system about the point O is $\left(\frac{Ma^2\Omega}{48}\right)n$. The value of n is ______.



Answer (49)

Sol. $L_s = L_{disc} + L_{rod}$

$$L_{\textbf{disc}} = \vec{r} \times \vec{p} + I_{\textbf{cm}} 4\Omega$$

$$= \frac{Ma^2}{32} \times 4\Omega + \frac{3a}{4} \times \frac{3a}{4} \times M\Omega$$

$$= \frac{11}{16} Ma^2\Omega$$

$$L_{rod} = \frac{Ma^2\Omega}{3}$$

$$L_{system} = \left(\frac{Ma^2}{3}\Omega + \frac{11}{16} Ma^2\Omega \right)$$

$$= \frac{49}{48} Ma^2\Omega$$

n = 49

19. A small object is placed at the center of a large evacuated hollow spherical container. Assume that the container is maintained at 0 K. At time t = 0, the temperature of the object is 200 K. The temperature of the object becomes 100 K at t = t_1 and 50 K at t = t_2. Assume the object and the container to be ideal black bodies. The heat capacity of the object does not depend on temperature. The ratio $\left(\frac{t_2}{t_1}\right)$ is ______.

Answer (9)

Sol. Heat radiated = $e\sigma AT^4$

$= KT^4$

$$-mS\frac{dT}{dt} = KT^4$$

$$-mS\int_{200}^{100} \frac{dT}{T^4} = Kt_1$$

$$t_1 = \frac{1}{K_1}\left[\frac{1}{100^3} - \frac{1}{200^3}\right] = \frac{1}{K_1}\left[\frac{7}{200^3}\right]$$

$$t_2 = \frac{1}{K_1}\left[\frac{1}{50^3} - \frac{1}{200^3}\right] = \frac{1}{K_1}\left[\frac{63}{200^3}\right]$$

$$\frac{t_2}{t_1} = 9$$

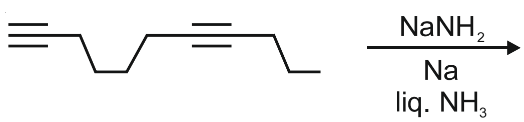
PART-II : CHEMISTRY

SECTION - 1

- This section contains **FOUR (04)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme:

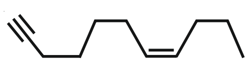
Full Marks	:	+3	If ONLY the correct option is chosen;
Zero Marks	:	0	If none of the options is chosen (i.e. the question is unanswered);
Negative Marks	:	–1	In all other cases.

1. The major product formed in the following reaction is



(A) 

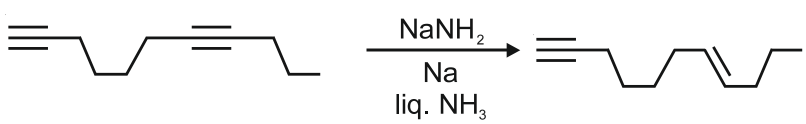
(B) 

(C) 

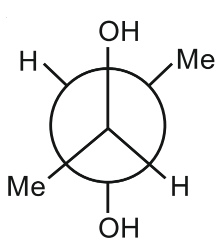
(D) 

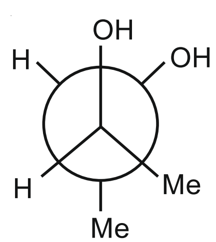
Answer (B)

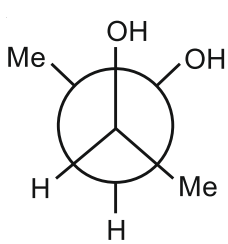
Sol. It is a case of Birch reduction. Alkynes on reaction with alkali metal in liq. NH_3 gives trans-alkene. But terminal alkynes do not get reduced.

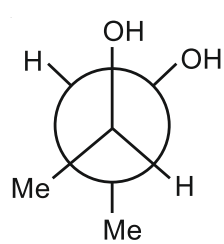


2. Among the following, the conformation that corresponds to the most stable conformation of *meso*-butane-2,3-diol is

(A) 

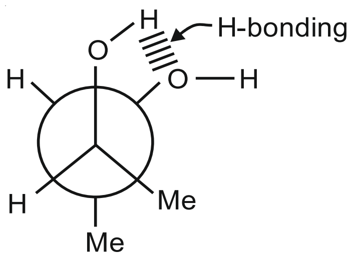
(B) 

(C) 

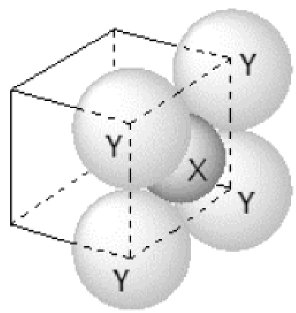
(D) 

Answer (B)

Sol. Meso compounds have plane of symmetry. In case of butan-2, 3-diol, gauche form is the most stable due to intra-molecular H-bonding.



3. For the given close packed structure of a salt made of cation X and anion Y shown below (ions of only one face are shown for clarity), the packing fraction is approximately (packing fraction = $\frac{\text{packing efficiency}}{100}$)



(A) 0.74
(B) 0.63
(C) 0.52
(D) 0.48

Answer (B)

Sol. a = edge length of unit cell

$2r_y = a$

$2(r_x + r_y) = \sqrt{2}a$

$2r_x + a = \sqrt{2}a$

$2r_x = a(\sqrt{2} - 1)$

$r_x = 0.207\ a$

Packing fraction = $\frac{3 \times \text{vol. of } x + \text{vol. of } y}{\text{vol. of unit cell}}$

$$= \frac{3 \times \frac{4}{3} \times \pi r_x^3 + \frac{4}{3} \times \pi \times r_y^3}{a^3}$$

$$= \frac{4 \times \pi \times (0.207a)^3 + \frac{4}{3} \times \pi \times (0.5a)^3}{a^3}$$

≈ 0.63

4. The calculated spin only magnetic moments of $[Cr(NH_3)_6]^{3+}$ and $[CuF_6]^{3-}$ in BM, respectively, are

(Atomic numbers of Cr and Cu are 24 and 29, respectively)

(A) 3.87 and 2.84
(B) 4.90 and 1.73
(C) 3.87 and 1.73
(D) 4.90 and 2.84

Answer (A)

Sol. $[Cr(NH_3)_6]^{3+} = Cr^{3+}$

$Cr^{3+} = 3d^3\ 4s^0$

It has 3 unpaired electrons

$\mu = \sqrt{n(n+2)}$ BM

$= \sqrt{3(3+2)}$ BM

= 3.87 BM

$[CuF_6]^{3-} = Cu^{+3}$

$Cu^{+3} = 3d^8\ 4s^0$

It has 2 unpaired electrons

$\mu = \sqrt{2(2+2)}$ BM

= 2.84 BM

SECTION - 2

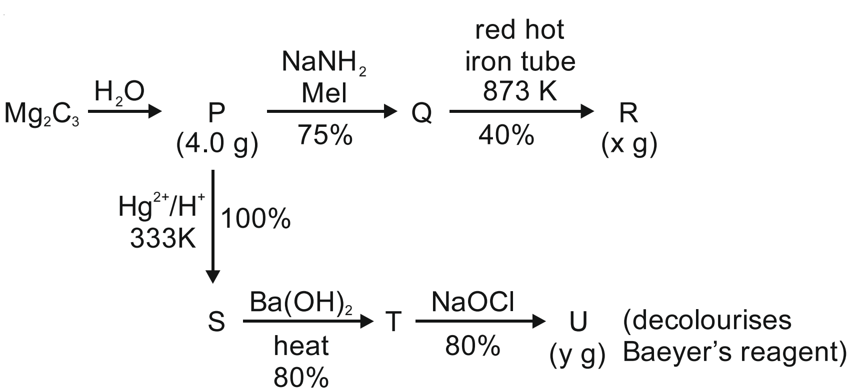
- This section contains **THREE (03)** question stems.
- There are **TWO (02)** questions corresponding to each question stem.
- The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical value corresponding to the answer in the designated place using the mouse and the on-screen virtual numeric keypad.
- If the numerical value has more than two decimal places, **truncate/round-off** the value to **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks	:	+2	If ONLY the correct numerical value is entered ate the designated place;
Zero Marks	:	0	In all other cases.

Question stem for Question Nos. 5 and 6

Question Stem

For the following reaction scheme, percentage yields are given along the arrow:



x g and y g are mass of R and U, respectively.

(Use : Molar mass (in g mol^{-1}) of H, C and O as 1, 12 and 16, respectively)

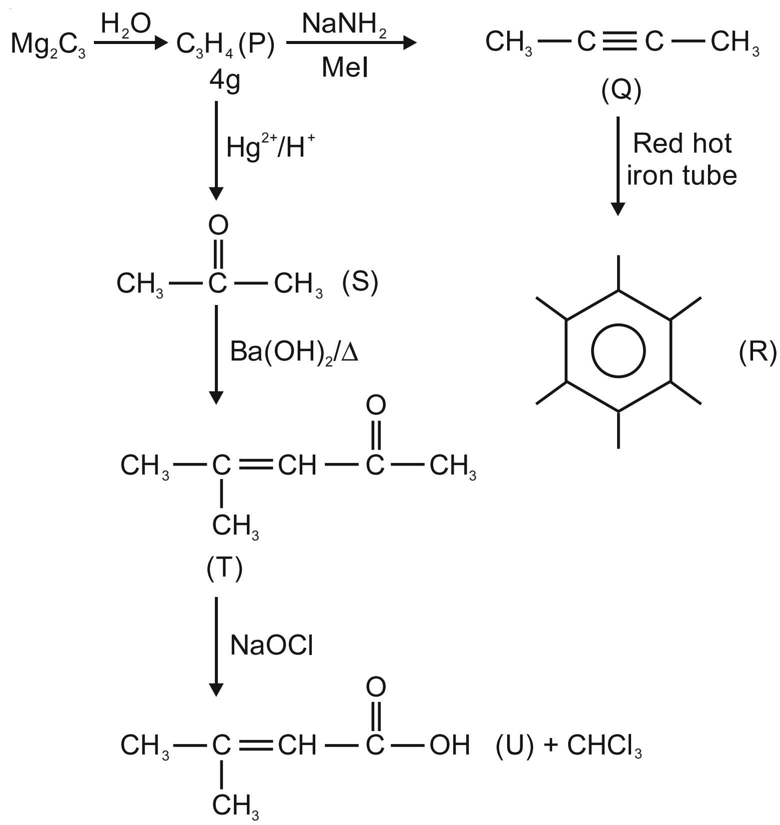
5. The value of x is ______.

Answer (1.62)

6. The value of y is ______.

Answer (3.20)

Sol. of Q. No. 5 and 6



4 g of C_3H_4 = 0.1 mol

From 0.1 mol of P, 0.01 mol of R will be produced

$\Rightarrow$ 1.62 g of R is produced

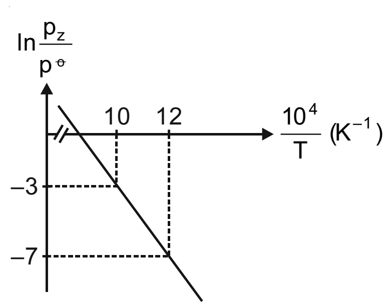
From 0.1 mol of P, 0.032 mol of U is produced

= 3.2 g of U is produced

Question stem for Question Nos. 7 and 8

Question Stem

For the reaction, $X(s) \rightleftharpoons Y(s) + Z(g)$, the plot of $\ln\frac{p_z}{p^{\ominus}}$ versus $\frac{10^4}{T}$ is given below (in solid line), where p_z is the pressure (in bar) of the gas Z at temperature T and $p^{\ominus} = 1$ bar.



(Given, $\frac{d(\ln K)}{d\left(\frac{1}{T}\right)} = -\frac{\Delta H^{\ominus}}{R}$, where the equilibrium constant, $K = \frac{p_z}{p^{\ominus}}$ and the gas constant, R = 8.314 J K^{-1} mol^{-1})

7. The value of standard enthalpy, $\Delta H^{\ominus}$ (in kJ mol–1) for the given reaction is ______.

Answer (166.28)

Sol. $X(s) \rightleftharpoons Y(s) + Z(g)$

Given $K = \frac{p_z}{p^{\ominus}}$

$$\ln K = \ln A - \frac{\Delta H^o}{RT}$$

$$\Rightarrow \ln\frac{p_z}{p^{\ominus}} = \ln A - \frac{\Delta H}{RT}$$

Slope of $\ln\frac{p_z}{p^{\ominus}}$ vs $\frac{1}{T}$ is $\frac{d\left[\ln\left(\frac{p_z}{p^{\ominus}}\right)\right]}{d\left(\frac{1}{T}\right)} = \frac{-\Delta H^o}{R}$

From the graph, we have $\frac{-\Delta H^o}{R} = -2\times10^4$

$\Rightarrow \Delta H^o = 2\times10^4 \times 8.314$ J

ΔH^o = 166.28 kJ mol^{-1}

8. The value of $\Delta S^{\ominus}$ (in J K^{-1} mol^{-1}) for the given reaction, at 1000 K is ____.

Answer (141.34)

Sol. $-RT\ln K = \Delta G^o = \Delta H^o - T\Delta S^o$

$$\ln k = -\frac{\Delta H^o}{RT} + \frac{\Delta S^o}{R}$$

$$\frac{\Delta S^o}{R} = 17$$

ΔS^o = 17R

= 141.338 J K^{-1}

Question stem for Question Nos. 9 and 10

Question Stem

The boiling point of water in a 0.1 molal silver intrate solution (soltuion A) is x°C. To this solution A, an equal volume of 0.1 molal aqueous barium chloride solution is added to make a new solution B. The difference in the boiling points of water in the two solutions A and B is y × 10^{-2} °C.

(Assume: Densities of the solutions A and B are the same as that of water and the soluble salts dissociate completely.

Use: Molal elevation constant (Ebullioscopic constant), K_b = 0.5 K kg mol^{-1}; Boiling point of pure water as 100°C.)

9. The value of x is ______.

Answer (100.1)

10. The value of |y| is _____.

Answer (2.5)

Sol. of Q. No. 9 and 10

Given molality of $AgNO_3$ solution is 0.1 molal (solution-A)

$\Delta T_b = ik_bm$

$AgNO_3 \rightarrow Ag^+ + NO_3^-$

van't Hoff factor (i) for $AgNO_3 = 2$

$\Delta T_b = 2 \times 0.5 \times 0.1$

$(T_s - T^o) = 0.1$

$(T_s)_A = 100.1°C$, so $x = 100.1$

Now solution-A of equal volume is mixed with 0.1 molal $BaCl_2$ solution to get solution-B. $AgNO_3$ reacts with $BaCl_2$ to form AgCl(s).

0.1 mole of $AgNO_3$ present in 1000 gram solvent or 1017 gram or 1017 mL solution,

milli moles of $AgNO_3$ in V ml 0.1 molal solution is nearly 0.1 V. Similarly in $BaCl_2$.

$2AgNO_3(aq) + BaCl_2(aq) \rightarrow 2AgCl(s) + Ba(NO_3)_2\,(aq)$

0.1 V	0.1 V	0	0
0	0.05 V	0.1 V	0.05 V

$$\Delta T_b = \left[\frac{0.05V\times 3}{2V} + \frac{0.05V\times 3}{2V}\right]\times 0.5 = 0.075$$

$$(T_s)_B = 100.075°C$$

$(T_s)_A - (T_s)_B = 100.1 - 100.075 = 0.025°C$

$= 2.5 \times 10^{-2}\,°C$

So $x = 100.1$ and $|y| = 2.5$

SECTION - 3

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options (A), (B), (C) & (D). **ONE OR MORE THAN ONE** of these four option(s) is(are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks	:	+4	If only (all) the correct option(s) is(are) chosen;
Partial Marks	:	+3	If all the four options are correct but ONLY three options are chosen;
Partial Marks	:	+2	If three or more options are correct but ONLY two options are chosen, both of which are correct;
Partial Marks	:	+1	If two or more options are correct but ONLY one option is chosen and it is a correct option;
Zero Marks	:	0	If none of the options is chosen (i.e. the question is unanswered);
Negative Marks	:	–2	In all other cases.

- For example, in a question, if (A), (B) and (D) are the ONLY three options corresponding to correct answers, then

  choosing ONLY (A), (B) and (D) will get +4 marks;

  choosing ONLY (A) and (B) will get +2 marks;

  choosing ONLY (A) and (D) will get +2 marks;

choosing ONLY (B) and (D) will get +2 marks;

choosing ONLY (A) will get +1 mark;

choosing ONLY (B) will get +1 mark;

choosing ONLY (D) will get +1 mark;

choosing no option(s) (i.e., the question is unanswered) will get 0 marks and

choosing any other options will get –2 marks.

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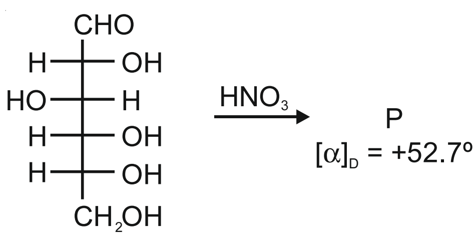
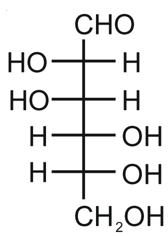
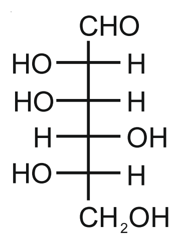
11. Given:

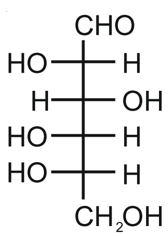


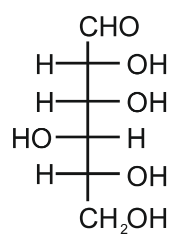
D-Glucose

The compound(s), which on reaction with HNO_3 will give the product having degree of rotation, $[\alpha]_D$ = –52.7° is(are)

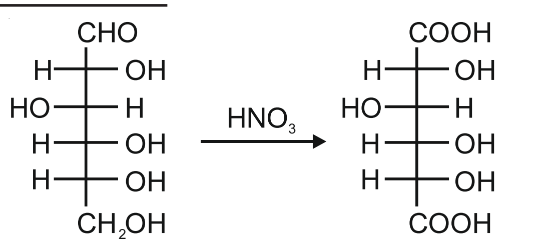
(A)


(B)


(C)


(D)


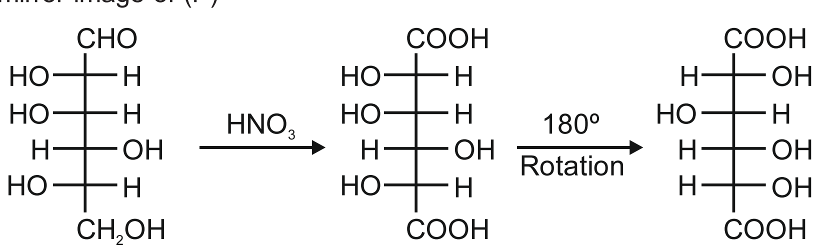
Answer (C, D)

Sol.


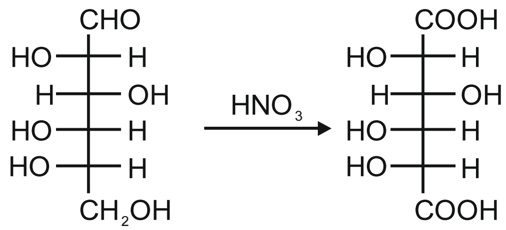
α = +52.7°

(P)

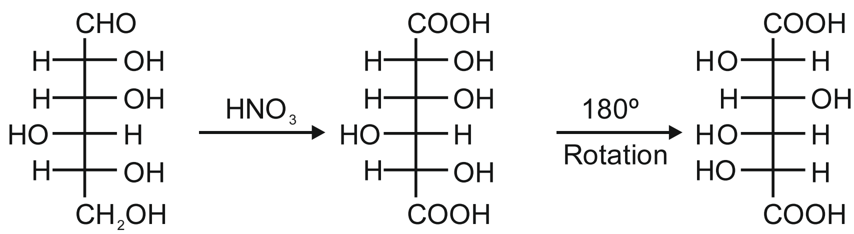
The enantiomer of (P) will have –52.7° rotation. So the reactant must be an isomer of D-glucose which can given the mirror image of (P)

(B)


Same as P

(C) 

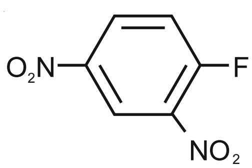
Enantiomer of (P)

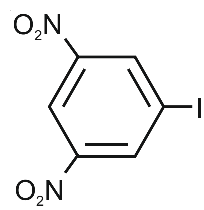
(D) 

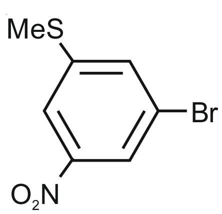
Enantiomer of (P)

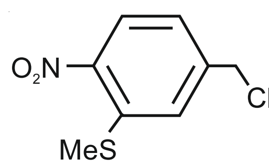
So answer must be C and D

12. The reaction of Q with PhSNa yields an organic compound (major product) that gives positive Carius test on treatment with Na_2O_2 followed by addition of $BaCl_2$. The correct option(s) for Q is(are)

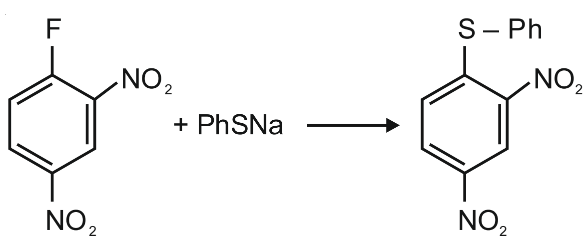
(A) 

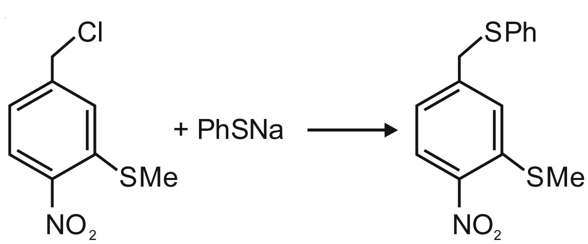
(B) 

(C) 

(D) 

Answer (A, D)

Sol. 



Answer should be (A) and (D)

Compounds given in option - B and C do not react with PhSNa.

13. The correct statement(s) related to colloids is(are)

(A) The process of precipitating colloidal sol by an electrolyte is called peptization

(B) Colloidal solution freezes at higher temperature than the true solution at the same concentration

(C) Surfactants form micelle above critical micelle concentration (CMC). CMC depends on temperature

(D) Micelles are macromolecular colloids

Answer (B, C)

Sol. Select the correct statements.

(A) The process of precipitating colloidal sol by an electrolyte is called peptization - False, (It is process of converting precipitate into colloid)

(B) Colloidal solution freezes at a higher temperature than the true solution at the same concentration - True (colligative properties)

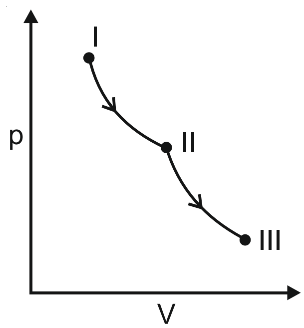
(C) Surfactants form miscelle above critical miscelle concentration (CMC). CMC depends on temperature - True

(D) Miscelles are macromolecular colloids - False, As misceles are associated colloids.

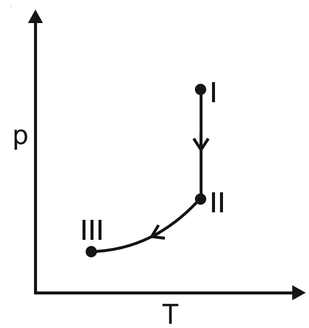
14. An ideal gas undergoes a reversible isothermal expansion from state I to state II followed by a reversible adiabatic expansion from state II to state III. The correct plot(s) representing the changes from state I to state III is(are)

(p: pressure, V: volume, T: temperature, H: enthalpy, S: entropy)

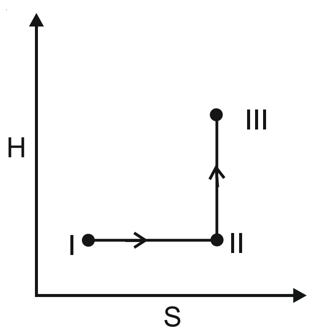
(A)



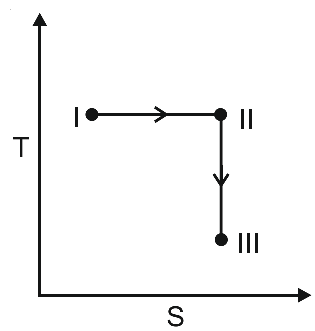
(B)



(C)



(D)



Answer (A, B, D)

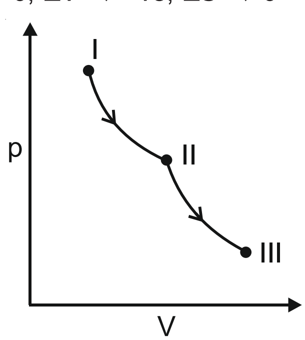
Sol. I $\rightarrow$ II $\rightarrow$ reversible, isothermal expansion,

T $\rightarrow$ constant, $\Delta V \rightarrow$ +ve, $\Delta S \rightarrow$ +ve $\Delta H \Rightarrow 0$

II $\rightarrow$ III $\rightarrow$ Reversible, adiabatic expansion

Q = 0, $\Delta V \rightarrow$ +ve, $\Delta S \rightarrow 0$

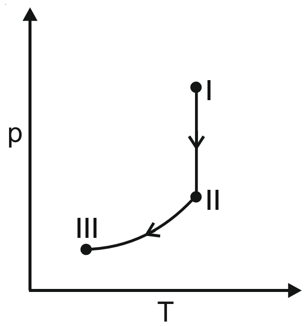
(A)



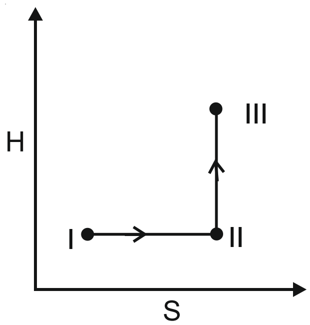
–ve slope - isothemal < adiabatic

I $\rightarrow$ II $\rightarrow$ Isothermal

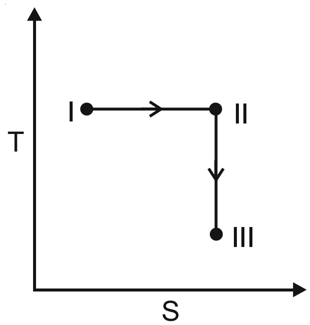
II $\rightarrow$ III $\rightarrow$ Adiabatic

(B) 

$I \rightarrow II \rightarrow T$ constant
$II \rightarrow III \rightarrow$ adiabatic

(C) 

$I \rightarrow II \rightarrow \Delta S \rightarrow +ve, \Delta H \Rightarrow 0$
$II \rightarrow III \rightarrow \Delta S \rightarrow 0, \Delta H \rightarrow -ve$

(D) 

$I \rightarrow II \rightarrow \Delta S \rightarrow +ve, \Delta T = 0$
$II \rightarrow III \rightarrow \Delta S \rightarrow 0$

15. The correct statement(s) related to the metal extraction processes is(are)

(A) A mixture of PbS and PbO undergoes self-reduction to produce Pb and SO_2.

(B) In the extraction process of copper from copper pyrites, silica is added to produce copper silicate

(C) Partial oxidation of sulphide ore of copper by roasting, followed by self-reduction produces blister copper

(D) In cyanide process, zinc powder is utilized to precipitate gold from $Na[Au(CN)_2]$

Answer (A, C, D)

Sol. $PbS + 2PbO \rightarrow 3Pb + SO_2$

Self reduction is taking place between PbS and PbO.

In the Bessemer converter : The raw material for the Bessemer converter is matte, i.e., Cu_2S + FeS (little). Here air blasting is initially done for slag formation and SiO_2 is added from external source.

$$FeS + \frac{3}{2}O_2 \rightarrow FeO + SO_2 \uparrow$$

$$SiO_2 + FeO \rightarrow FeSiO_3 \text{ (slag)}$$

During slag formation, the characteristic green flame is observed at the mouth of the Bessemer converter which indicates the presence of iron in the form of FeO. Disappearance of this green flame indicates that the slag formation is complete. Then air blasting is stopped and slag is removed.

Again air blasting is restarted for partial roasting before self reduction, until two-thirds of Cu_2S is converted into Cu_2O. After this, only heating is continued for the self reduction process.

$$Cu_2S + \frac{3}{2}O_2 \rightarrow Cu_2O + SO_2 \uparrow$$

$$Cu_2S + 2Cu_2O \rightarrow 6Cu(l) + SO_2 \uparrow \qquad \text{(self reduction)}$$

and $Cu_2S + 2O_2 \rightarrow Cu_2SO_4$

$$Cu_2S + Cu_2SO_4 \rightarrow 4Cu + 2SO_2 \uparrow \qquad \text{(self reduction)}$$

Thus the molten Cu obtained is poured into large container and allowed to cool and during cooling the dissolved SO_2 comes up to the surface and forms blisters. It is known as blister copper.

$$2Na[Au(CN)_2] + Zn \rightarrow Na_2[Zn(CN)_4] + 2Au \downarrow$$

16. A mixture of two salts is used to prepare a solution S, which gives the following results:

$$\begin{matrix}\text{White} \\ \text{precipitate(s)} \\ \text{only}\end{matrix} \xleftarrow[\text{Room temperature}]{\text{Dilute NaOH(aq)}} \begin{matrix}\text{S} \\ \text{(aq solution} \\ \text{of the salts)}\end{matrix} \xrightarrow[\text{Room temperature}]{\text{Dilute HCl(aq)}} \begin{matrix}\text{White} \\ \text{precipitate(s)} \\ \text{only}\end{matrix}$$

The correct option(s) for the salt mixture is(are)

(A) $Pb(NO_3)_2$ and $Zn(NO_3)_2$

(B) $Pb(NO_3)_2$ and $Bi(NO_3)_3$

(C) $AgNO_3$ and $Bi(NO_3)_3$

(D) $Pb(NO_3)_2$ and $Hg(NO_3)_2$

Answer (A, B)

Sol. $Pb(NO_3)_2 \xrightarrow[\text{Room temp.}]{\text{dil HCl}} \underset{\text{(white ppt)}}{PbCl_2}$

$$Pb(NO_3)_2 \xrightarrow[\text{Room temperature}]{\text{Dilute NaOH (aq)}} \underset{\text{(white ppt)}}{Pb(OH)_2}$$

$$Zn(NO_3)_2 \xrightarrow[\text{Room temperature}]{\text{dil HCl}} \underset{\text{(soluble)}}{Zn^{2+} + 2Cl^-}$$

$$Zn(NO_3)_2 \xrightarrow[\text{Room temperature}]{\text{dil NaOH(aq)}} \underset{\text{(white ppt)}}{Zn(OH)_2}$$

$$Bi(NO_3)_3 \xrightarrow[\text{Room temperature}]{\text{dil HCl(aq)}} \underset{\text{(White ppt)}}{BiOCl}$$

$$Bi(NO_3)_3 \xrightarrow[\text{Room temperature}]{\text{dil NaOH(aq)}} \underset{\text{(White ppt)}}{Bi(OH)_3}$$

$$AgNO_3 \xrightarrow[\text{Room temperature}]{\text{dil HCl}} \underset{\text{(White ppt)}}{AgCl}$$

$$AgNO_3 \xrightarrow[\text{Room temperature}]{\text{dilute NaOH(aq)}} \underset{\text{(Brownish black ppt)}}{Ag_2O}$$

$$Hg(NO_3)_2 \xrightarrow[\text{Room temperature}]{\text{dil HCl}} \underset{\text{(soluble)}}{Hg^{2+} + 2Cl^-}$$

$$Hg(NO_3)_2 \xrightarrow[\text{Room temperature}]{\text{dilute NaOH(aq)}} \underset{\text{(Yellow precipitate)}}{HgO}$$

SECTION - 4

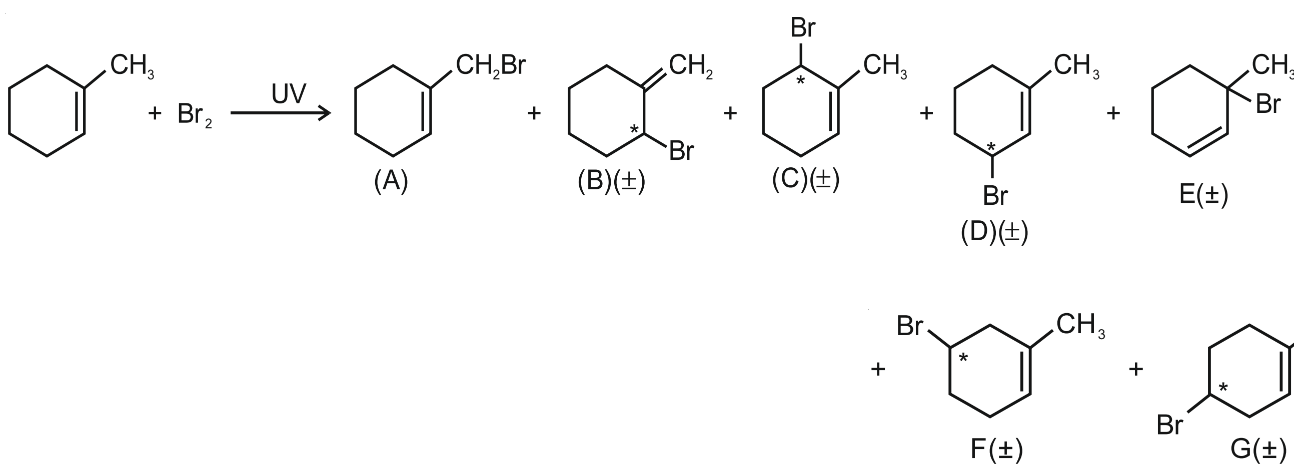
- This section contains **THREE (03)** questions.
- The answer to each question is a NON-NEGATIVE INTEGER.
- For each question, enter the correct integer corresponding to the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks	:	+4	If ONLY the correct numerical value is entered.
Zero Marks	:	0	In all other cases.

17. The maximum number of possible isomers (including stereoisomers) which may be formed on *mono*-bromination of 1-methylcyclohex-1-ene using Br_2 and UV light is ______.

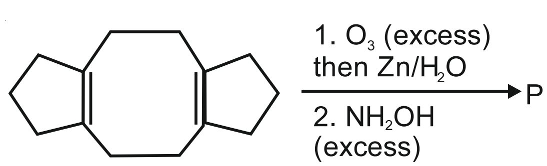
Answer (13)

Sol.



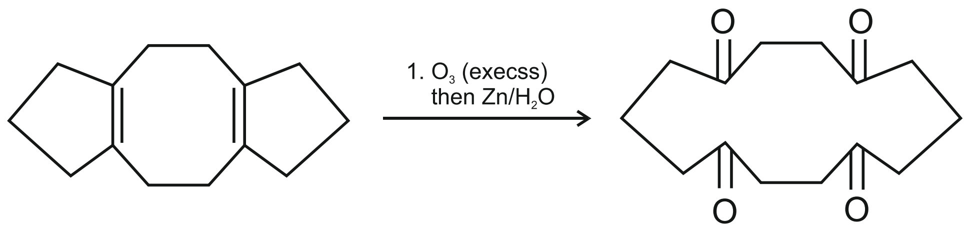
Monobromination of 1-methylcyclohexene in presence of UV light proceeds by free radical mechanism. The allyl radicals are formed which are stabilised by resonance. The secondary alkyl radicals are also formed which are stabilised by hyperconjugation. Of the seven products formed, six of them are optically active. So, 13 possible isomers are formed.

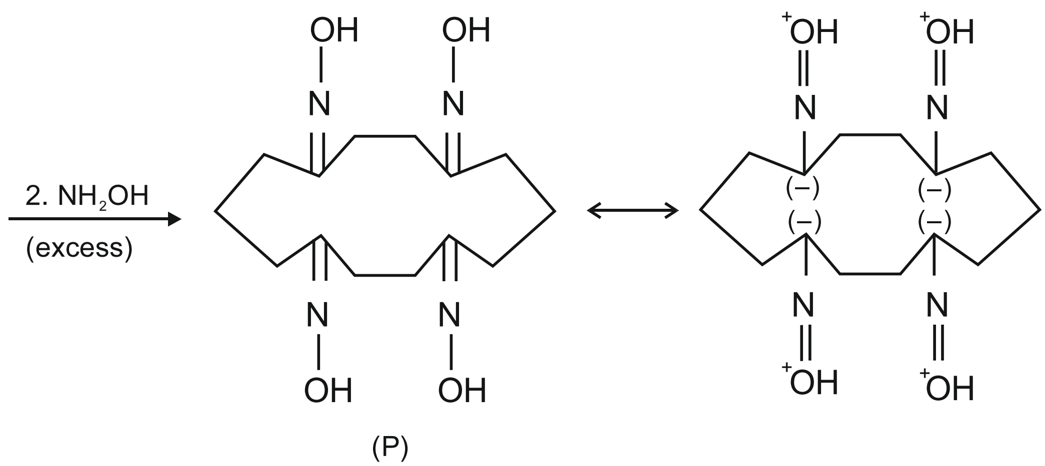
18. In the reaction given below, the total number of atoms having sp^2 hybridization in the major product P is ______.



Answer (12)

Sol.





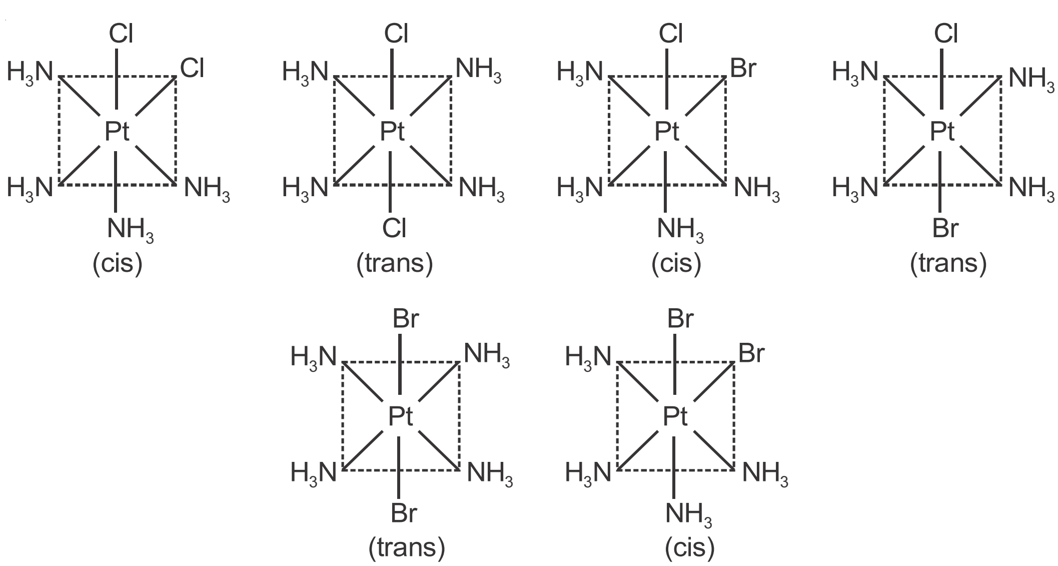
The total number of atoms having sp^2 hybridisation in the major product (P) = 12

This includes 4 C-atoms, 4 N-atoms and 4 O-atoms.

19. The total number of possible isomers for $[Pt(NH_3)_4Cl_2]Br_2$ is

Answer (6)

Sol. The given complex $[Pt(NH_3)_4Cl_2]Br_2$ has three ionisation isomers and each of them has two geometrical isomers.



PART-III : MATHEMATICS

SECTION - 1

- This section contains **FOUR (04)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +3 If ONLY the correct option is chosen;

Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered);

Negative Marks : –1 In all other cases.

1. Consider a triangle Δ whose two sides lie on the x-axis and the line $x + y + 1 = 0$. If the orthocentre of Δ is (1, 1), then the equation of the circle passing through the vertices of the triangle Δ is

(A) $x^2 + y^2 - 3x + y = 0$ (B) $x^2 + y^2 + x + 3y = 0$

(C) $x^2 + y^2 + 2y - 1 = 0$ (D) $x^2 + y^2 + x + y = 0$

Answer (B)

Sol. As we know mirror image of orthocentre lie on circumcircle.

Image of (1, 1) in x-axis is (1, –1)

Image of (1, 1) in $x + y + 1 = 0$ is (–2, –2).

$\therefore$ The required circle will be passing through both (1, –1) and (–2, –2).

$\therefore$ Only $x^2 + y^2 + x + 3y = 0$ satisfy both.

2. The area of the region

$$\left\{(x, y): 0 \le x \le \frac{9}{4}, \quad 0 \le y \le 1, \quad x \ge 3y, \quad x + y \ge 2\right\}$$

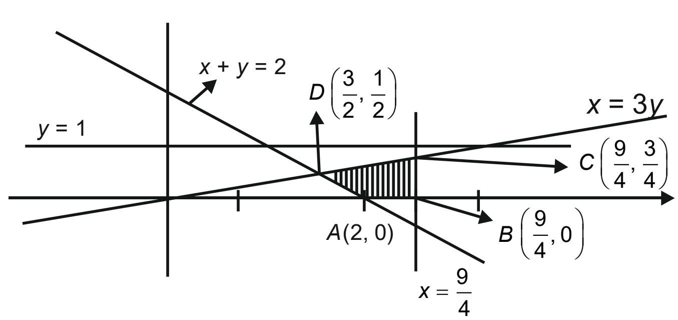
is

(A) $\frac{11}{32}$ (B) $\frac{35}{96}$

(C) $\frac{37}{96}$ (D) $\frac{13}{32}$

Answer (A)

Sol. Rough sketch of required region is



$\therefore$ Required area is

Area of ΔACD + Area of ΔABC

i.e., $\frac{1}{4} + \frac{3}{32} = \frac{11}{32}$ sq. units

3. Consider three sets $E_1 = \{1, 2, 3\}$, $F_1 = \{1, 3, 4\}$ and $G_1 = \{2, 3, 4, 5\}$. Two elements are chosen at random, without replacement, from the set E_1, and let S_1 denote the set of these chosen elements. Let $E_2 = E_1 - S_1$ and $F_2 = F_1 \cup S_1$. Now two elements are chosen at random, without replacement, from the set F_2 and let S_2 dentoe the set of these chosen elements.

Let $G_2 = G_1 \cup S_2$. Finally, two elements are chosen at random, without replacement from the set G_2 and let S_3 denote the set of these chosen elements.

Let $E_3 = E_2 \cup S_3$. Given that $E_1 = E_3$, let p be the conditional probability of the event $S_1 = \{1, 2\}$. Then the value of p is

(A) $\frac{1}{5}$

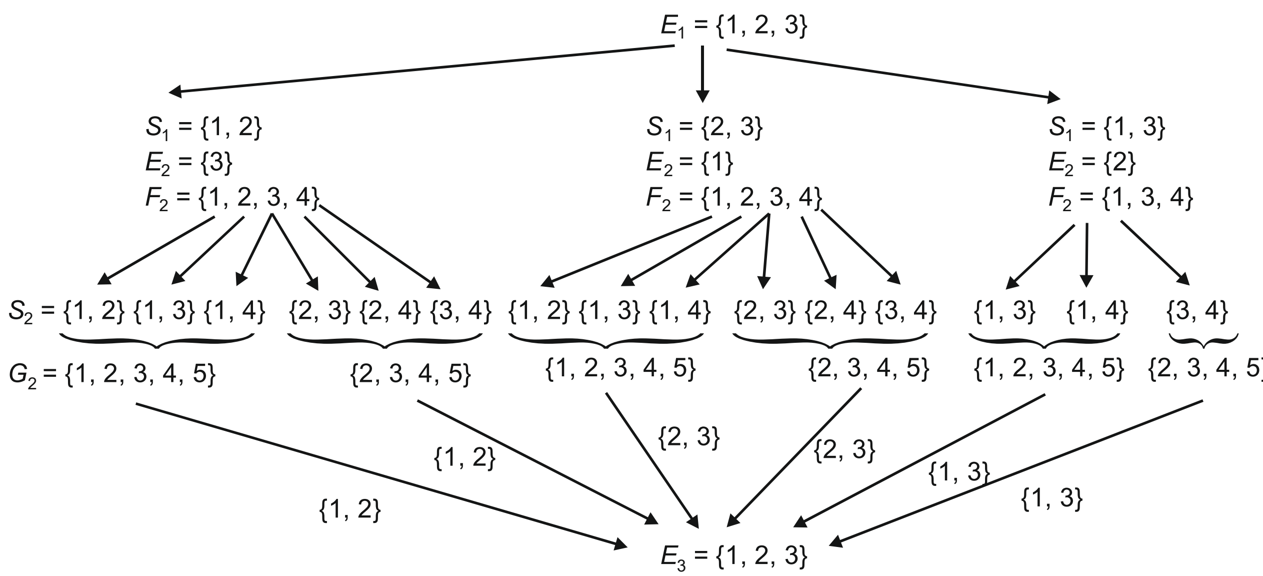
(B) $\frac{3}{5}$

(C) $\frac{1}{2}$

(D) $\frac{2}{5}$

Answer (A)

Sol. We will follow the tree diagram,



$$P(E_1 = E_3) = \frac{1}{3}\left[\frac{1}{2}\times\frac{1}{10} + \frac{1}{2}\times 0 + \frac{1}{2}\times\frac{1}{10} + \frac{1}{2}\times\frac{1}{6} + \frac{2}{3}\times\frac{1}{10} + \frac{1}{3}\times 0\right]$$

$$= \frac{1}{3}\left[\frac{1}{4}\right]$$

$$\text{Required probability} = \frac{\frac{1}{3}\left[\frac{1}{2}\times\frac{1}{10}\right]}{\frac{1}{3}\times\frac{1}{4}} = \frac{1}{5}$$

4. Let $\theta_1, \theta_2, ..., \theta_{10}$ be positive valued angles (in radian) such that $\theta_1 + \theta_2 + ... + \theta_{10} = 2\pi$. Define the complex numbers $z_1 = e^{i\theta_1}$, $z_k = z_{k-1}e^{i\theta_k}$ for $k = 2, 3, ..., 10$, where $i = \sqrt{-1}$. Consider the statement P and Q given below:

$$P : |z_2 - z_1| + |z_3 - z_2| + ... + |z_{10} - z_9| + |z_1 - z_{10}| \le 2\pi$$

$$Q : |z_2^2 - z_1^2| + |z_3^2 - z_2^2| + ... + |z_{10}^2 - z_9^2| + |z_1^2 - z_{10}^2| \le 4\pi$$

Then,

(A) P is **TRUE** and Q is **FALSE**

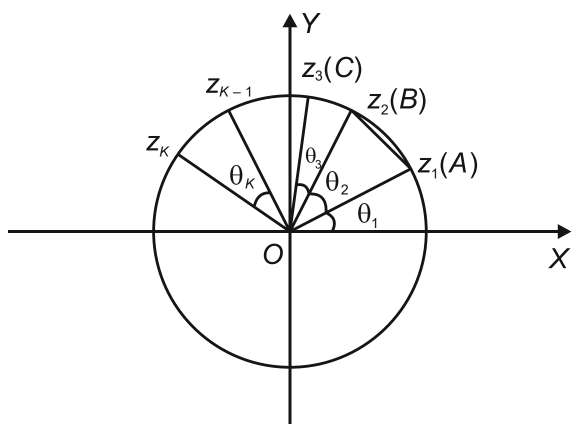
(B) Q is **TRUE** and P is **FALSE**

(C) Both P and Q are **TRUE**

(D) Both P and Q are **FALSE**

Answer (C)

Sol.



$|z_2 - z_1|$ = length of line $AB \le$ length of arc AB

$|z_3 - z_2|$ = length of line $BC \le$ length of arc BC

$\therefore$ Sum of length of these 10 lines $\le$ Sum of length of arcs (i.e. 2π)

(As $(\theta_1 + \theta_2 + ... + \theta_{10}) = 2\pi$)

$$\therefore \quad |z_2 - z_1| + |z_3 - z_2| + ... + |z_1 - z_{10}| \le 2\pi$$

And $|z_k^2 - z_{k-1}^2| = |z_k - z_{k-1}|\,|z_k + z_{k-1}|$

As we know $|z_k + z_{k-1}| \le |z_k| + |z_{k-1}| \le 2$

$$|z_2^2 - z_1^2| + |z_3^2 - z_2^2| + ... + |z_1^2 - z_{10}^2| \le 2\left(|z_2 - z_1| + |z_3 - z_2| + ... + |z_1 - z_{10}|\right)$$

$$\le 2\,(2\pi)$$

$\therefore$ Both (P) and (Q) are true.

SECTION - 2

- This section contains **THREE (03)** question stems.
- There are **TWO (02)** questions corresponding to each question stem.
- The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical value corresponding to the answer in the designated place using the mouse and the on-screen virtual numerical keypad.
- If the numerical value has more than two decimal places, **truncate/round-off** the value to **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +2 If ONLY the correct numerical value is entered at the designated place;

Zero Marks : 0 In all other cases.

Question Stem for Question Nos. 5 and 6

Question Stem

Three numbers are chosen at random, one after another with replacement, from the set S = {1, 2, 3, ..., 100}. Let p_1 be the probability that the maximum of chosen numbers is at least 81 and p_2 be the probability that the minimum of chosen numbers is at most 40.

5. The value of $\frac{625}{4}p_1$ is _____.

Answer (76.25)

Sol. For p_1, we need to remove the cases when all three numbers are less than or equal to 80.

So, $p_1 = 1 - \left(\frac{80}{100}\right)^3 = \frac{61}{125}$

So, $\frac{625}{4}p_1 = \frac{625}{4} \times \frac{61}{125} = \frac{305}{4} = 76.25$

6. The value of $\frac{125}{4}p_2$ is _____.

Answer (24.50)

Sol. For p_2, we need to remove the cases when all three numbers are greater than 40.

So, $p_2 = 1 - \left(\frac{60}{100}\right)^3 = \frac{98}{125}$

So, $\frac{125}{4}p_2 = \frac{125}{4} \times \frac{98}{125} = 24.50$

Question Stem for Question Nos. 7 and 8

Question Stem

Let α, β and γ be real numbers such that the system of linear equations

$x + 2y + 3z = \alpha$

$4x + 5y + 6z = \beta$

$7x + 8y + 9z = \gamma - 1$

is consistent. Let $|M|$ represent the determinant of the matrix

$$M = \begin{bmatrix} \alpha & 2 & \gamma \\ \beta & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Let P be the plane containing all those (α, β, γ) for which the above system of linear equations is consistent, and D be the **square** of the distance of the point (0, 1, 0) from the plane P.

7. The value of $|M|$ is _________.

Answer (1)

8. The value of D is _____.

Answer (1.50)

Sol. Solution for Q 7 and 8

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0$$

Given system of equation will be consistent even if $\alpha = \beta = \gamma - 1 = 0$, i.e. equations will form homogeneous system.

So, $\alpha = 0$, $\beta = 0$, $\gamma = 1$

$$M = \begin{vmatrix} 0 & 2 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = -1(-1) = +1$$

As given equations are consistent

$x + 2y + 3z - \alpha = 0$ ...P_1

$4x + 5y + 6z - \beta = 0$ ...P_2

$7x + 8y + 9z - (\gamma - 1) = 0$ ...P_3

For some scalar λ and μ

$\mu P_1 + \lambda P_2 = P_3$

$\mu(x + 2y + 3z - \alpha) + \lambda(4x + 5y + 6z - \beta) = 7x + 8y + 9z - (\gamma - 1)$

Comparing coefficients

$\mu + 4\lambda = 7$, $2\mu + 5\lambda = 8$, $3\mu + 6\lambda = 9$

$\lambda = 2$ and $\mu = -1$ satisfy all these conditions

comparing constant terms,

$-\alpha\mu - \beta\lambda = -(\gamma - 1)$

$\alpha - 2\beta + \gamma = 1$

So equation of plane is

$x - 2y + z = 1$

Distance from (0, 1, 0) = $\left|\frac{-2-1}{\sqrt{6}}\right| = \frac{3}{\sqrt{6}}$

$$D = \left(\frac{3}{\sqrt{6}}\right)^2 = \frac{3}{2} = 1.50$$

Question Stem for Question Nos. 9 and 10

Question Stem

Consider the lines L_1 and L_2 defined by

$L_1 : x\sqrt{2} + y - 1 = 0$ and $L_2 : x\sqrt{2} - y + 1 = 0$

For a fixed constant λ, let C be the locus of a point P such that the product of the distance of P from L_1 and the distance of P from L_2 is λ^2. The line $y = 2x + 1$ meets C at two points R and S, where the distance between R and S is $\sqrt{270}$.

Let the perpendicular bisector of RS meet C at two distinct points R' and S'. Let D be the square of the distance between R' and S'.

9. The value of λ^2 is ________.

Answer (9)

10. The value of D is _____.

Answer (77.14)

Sol. Solution for Q 9 and 10

$$C : \left|\frac{x\sqrt{2} + y - 1}{\sqrt{3}}\right| \left|\frac{x\sqrt{2} - y + 1}{\sqrt{3}}\right| = \lambda^2$$

$$\Rightarrow C : \left|2x^2 - (y-1)^2\right| = 3\lambda^2$$

C cuts $y - 1 = 2x$ at $R(x_1, y_1)$ and $S(x_2, y_2)$

So, $\left|2x^2 - 4x^2\right| = 3\lambda^2 \quad \Rightarrow \quad x = \pm\sqrt{\frac{3}{2}}\,|\lambda|$

So, $|x_1 - x_2| = \sqrt{6}\,|\lambda|$ and $|y_1 - y_2| = 2|x_1 - x_2| = 2\sqrt{6}\,|\lambda|$

$\because$ $RS^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 \Rightarrow 270 = 30\lambda^2 \Rightarrow \lambda^2 = 9$

$\because$ Slope of $RS = 2$ and mid-point of RS is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \equiv (0, 1)$

So, $R'S' \equiv y - 1 = -\frac{1}{2}x$

Solving $y - 1 = -\frac{1}{2}x$ with 'C' we get $x^2 = \frac{12}{7}\lambda^2$

$$\Rightarrow |x_1 - x_2| = 2\sqrt{\frac{12}{7}}\,|\lambda| \text{ and } |y_1 - y_2| = \frac{1}{2}|x_1 - x_2| = \sqrt{\frac{12}{7}}\,|\lambda|$$

Hence, $D = (R'S')^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 = \frac{12}{7}.9 \times 5 \approx 77.14$

SECTION - 3

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONE OR MORE THAN ONE** of these four option(s) is(are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks	:	+4	If only (all) the correct option(s) is(are) chosen;
Partial Marks	:	+3	If all the four options are correct but ONLY three options are chosen;
Partial Marks	:	+2	If three or more options are correct but ONLY two options are chosen, both of which are correct;
Partial Marks	:	+1	If two or more options are correct but ONLY one option is chosen and it is a correct option;
Zero Marks	:	0	If unanswere;
Negative Marks	:	–2	In all other cases.

- For example, in a question, if (A), (B) and (D) are the ONLY three options corresponding to correct answers, then

  choosing ONLY (A), (B) and (D) will get +4 marks;

  choosing ONLY (A) and (B) will get +2 marks;

  choosing ONLY (A) and (D) will get +2 marks;

  choosing ONLY (B) and (D) will get +2 marks;

  choosing ONLY (A) will get +1 mark;

  choosing ONLY (B) will get +1 mark;

  choosing ONLY (D) will get +1 mark;

  choosing no option(s) (i.e., the question is unanswered) will get 0 marks and

  choosing any other option(s) will get –2 marks.

11. For any 3 × 3 matrix M, let $|M|$ denote the determinant of M. Let

$$E = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 8 & 13 & 18 \end{bmatrix}, \ P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } F = \begin{bmatrix} 1 & 3 & 2 \\ 8 & 18 & 13 \\ 2 & 4 & 3 \end{bmatrix}$$

If Q is a nonsingular matrix of order 3 × 3, then which of the following statements is(are) **TRUE**?

(A) $F = PEP$ and $P^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(B) $\left|EQ + PFQ^{-1}\right| = |EQ| + \left|PFQ^{-1}\right|$

(C) $\left|(EF)^3\right| > |EF|^2$

(D) Sum of the diagonal entries of $P^{-1}EP + F$ is equal to the sum of diagonal entries of $E + P^{-1}FP$

Answer (A, B, D)

Sol. $\because$ P is formed from I by exchanging second and third row or by exchanging second and third column.

So, PA is a matrix formed from A by changing second and third row.

Similarly AP is a matrix formed from A by changing second and third column.

Hence, Tr(PAP) = Tr(A) ...(1)

(A) Clearly, $P.P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1$

and $PE = \begin{bmatrix} 1 & 2 & 3 \\ 8 & 13 & 18 \\ 2 & 3 & 4 \end{bmatrix} \Rightarrow PEP = \begin{bmatrix} 1 & 3 & 2 \\ 8 & 18 & 13 \\ 2 & 4 & 3 \end{bmatrix} = F$

$\Rightarrow$ $PEP = F \Rightarrow PFP = E$ ...(2)

(B) $\because |E| = |F| = 0$

So, $|EQ + PFQ^{-1}| = |PFPQ + PFQ^{-1}| = |P|\,|F|\,|PQ + Q^{-1}| = 0$

Also, $|EQ| + |PFQ^{-1}| = 0$

(C) From (2); $PFP = E$ and $|P| = -1$

So, $|F| = |E|$

Also, $|E| = 0 = |F|$

So, $|EF|^3 = 0 = |EF|^2$

(D) $\because P^2 = I \;\Rightarrow\; P^{-1} = P$

So, $\text{Tr}\left(P^{-1}EP + F\right) = \text{Tr}(PEP + F) = \text{Tr}(2F)$

Also $\text{Tr}\left(E + P^{-1}FP\right) = \text{Tr}(E + PFP) = \text{Tr}(2E)$

Given that Tr(E) = Tr(F)

$\Rightarrow$ Tr($2E$) = Tr($2F$)

12. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \frac{x^2 - 3x - 6}{x^2 + 2x + 4}$$

Then which of the following statements is (are) **TRUE**?

(A) f is decreasing in the interval (–2, –1)

(B) f is increasing in the interval (1, 2)

(C) f is onto

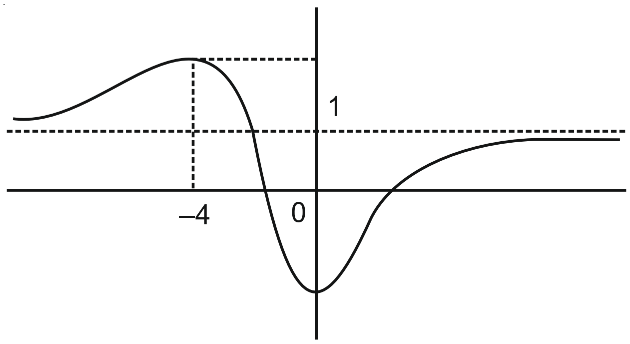
(D) Range of f is $\left[-\frac{3}{2}, 2\right]$

Answer (A, B)

Sol. $f(x) = \dfrac{x^2 - 3x - 6}{x^2 + 2x + 4}$

$\Rightarrow$ $f'(x) = \dfrac{5x(x+4)}{(x^2 + 2x + 4)^2}$

$\Rightarrow$ $f(x)$ has local maxima at $x = -4$ and minima at $x = 0$



Range of $f(x)$ is $\left[-\frac{3}{2}, \frac{11}{6}\right]$

13. Let E, F and G be three events having probabilities

$P(E) = \frac{1}{8}, P(F) = \frac{1}{6}$ and $P(G) = \frac{1}{4}$, and $P(E \cap F \cap G) = \frac{1}{10}$.

For any event H, if H^c denotes its complement, then which of the following statements is(are) **TRUE**?

(A) $P(E \cap F \cap G^c) \le \frac{1}{40}$

(B) $P(E^c \cap F \cap G) \le \frac{1}{15}$

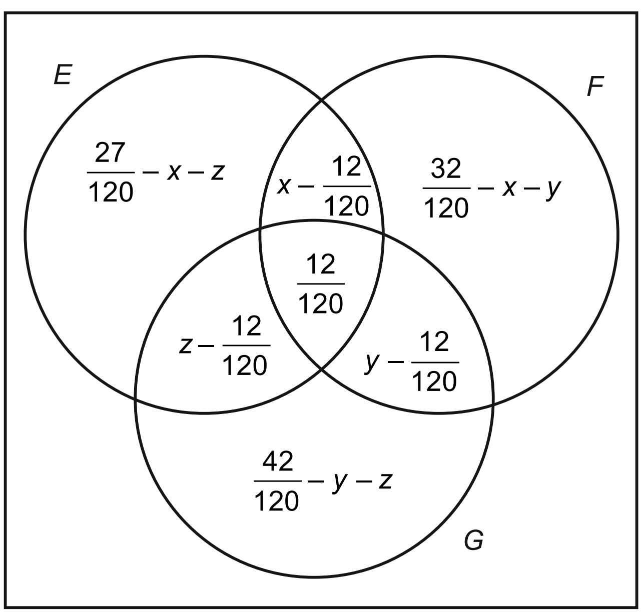
(C) $P(E \cup F \cup G) \le \frac{13}{24}$

(D) $P(E^c \cap F^c \cap G^c) \le \frac{5}{12}$

Answer (A, B, C)

Sol. Let $P(E \cap F) = x$, $P(F \cap G) = y$ and $P(E \cap G) = z$

Clearly $x, y, z \ge \frac{1}{10}$



$\because x + z \le \frac{27}{120} \Rightarrow x, z \le \frac{15}{120}$

$x + y \le \frac{32}{120} \Rightarrow x, y \le \frac{20}{120}$

and $y+z \le \frac{42}{120} \Rightarrow y, z \le \frac{30}{120}$

Now $P(E \cap F \cap G^c) = x - \frac{12}{120} \le \frac{3}{120} = \frac{1}{40}$

$P(E^c \cap F \cap G) = y - \frac{12}{120} \le \frac{80}{120} = \frac{1}{15}$

$P(E \cup F \cup G) \le P(E) + P(F) + P(G) = \frac{13}{24}$

and $P(E^c \cap F^c \cap G^c) = 1 - P(E \cup F \cup G) \ge \frac{11}{24} \ge \frac{5}{12}$

14. For any 3 × 3 matrix M, let $|M|$ denote the determinant of M. Let I be the 3 × 3 identify matrix. Let E and F be two 3 × 3 matrices such that $(I - EF)$ is invertible. If $G = (I - EF)^{-1}$, then which of the following statements is (are) **TRUE**?

(A) $|FE| = |I - FE|\,|FGE|$

(B) $(I - FE)(I + FGE) = I$

(C) $EFG = GEF$

(D) $(I - FE)(I - FGE) = I$

Answer (A, B, C)

Sol. $\because$ $I - EF = G^{-1}$

$\Rightarrow$ $G - GEF = I$ …(1)

and $G - EFG = I$ …(2)

Clearly $GEF = EFG$ (option C is correct)

Also $(I - FE)(I + FGE) = I - FE + FGE - FE + FGE$

$= I - FE + FGE - F(G - I)E$

$= I - FE + FGE - FGE + FE$

$= I$ (option B is correct and D is incorrect)

Now, $(I - FE)(I - FGE) = I - FE - FGE + F(G - I)E$

$= I - 2FE$

$\Rightarrow$ $(I - FE)(- FGE) = - FE$

$\Rightarrow$ $|I - FE||FGE| = |FE|$

15. For any positive integer n, let $S_n : (0, \infty) \to \mathbb{R}$ be defined by

$$S_n(x) = \sum_{k=1}^{n} \cot^{-1}\left(\frac{1 + k(k+1)x^2}{x}\right)$$

where for any $x \in \mathbb{R}$, $\cot^{-1}(x) \in (0, \pi)$ and $\tan^{-1}(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then which of the following statements is (are) **TRUE**?

(A) $S_{10}(x) = \frac{\pi}{2} - \tan^{-1}\left(\frac{1+11x^2}{10x}\right)$, for all $x > 0$

(B) $\lim\limits_{n \to \infty} \cot(S_n(x)) = x$, for all $x > 0$

(C) The equation $S_3(x) = \frac{\pi}{4}$ has a root in $(0, \infty)$

(D) $\tan(S_n(x)) \le \frac{1}{2}$, for all $n \ge 1$ and $x > 0$

Answer (A, B)

Sol. $S_n(x) = \sum_{k=1}^{n} \tan^{-1}\left(\frac{(k+1)x - kx}{1+kx.(k+1)x}\right)$

$$= \sum_{k=1}^{n}\left(\tan^{-1}(k+1)x - \tan^{-1} kx\right)$$

$$= \tan^{-1}(n+1)x - \tan^{-1} x = \tan^{-1}\left(\frac{nx}{1+(n+1)x^2}\right)$$

Now (A) $S_{10}(x) = \tan^{-1}\left(\frac{10x}{1+11x^2}\right) = \frac{\pi}{2} - \tan^{-1}\left(\frac{1+11x^2}{10x}\right)$

(B) $\lim_{n\to\infty} \cot(S_n(x)) = \cot\left(\tan^{-1}\left(\frac{x}{x^2}\right)\right) = x$

(C) $S_3(x) = \frac{\pi}{4} \Rightarrow \frac{3x}{1+4x^2} = 1 \Rightarrow 4x^2 - 3x + 1 = 0$ has no real root.

(D) For $x = 1$, $\tan(S_n(x)) = \frac{n}{n+2}$ which is greater than $\frac{1}{2}$ for $n \geq 3$ so this option is incorrect.

16. For any complex number $w = c + id$, let $\arg(w) \in (-\pi, \pi]$, where $i = \sqrt{-1}$. Let α and β be real numbers such that for all complex numbers $z = x + iy$ satisfying $\arg\left(\frac{z+\alpha}{z+\beta}\right) = \frac{\pi}{4}$, the ordered pair (x, y) lies on the circle

$x^2 + y^2 + 5x - 3y + 4 = 0$

Then which of the following statements is (are) **TRUE**?

(A) $\alpha = -1$

(B) $\alpha\beta = 4$

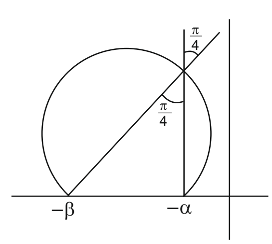
(C) $\alpha\beta = -4$

(D) $\beta = 4$

Answer (B, D)

Sol. Circle $x^2 + y^2 + 5x - 3y + 4 = 0$ cuts the real axis (x-axis) at $(-4, 0)$, $(-1, 0)$

Clearly $\alpha = 1$ and $\beta = 4$



SECTION - 4

- This section contains **THREE (03)** questions.
- The answer to each question is a **NON-NEGATIVE INTEGER**.
- For each question, enter the correct integer corresponding to the answer using the moust and the on-screen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks	:	+4	If ONLY the correct integer is entered.
Zero Marks	:	0	In all other cases.

17. For $x \in \mathbb{R}$, the number of real roots of the equation

$$3x^2 - 4|x^2 - 1| + x - 1 = 0$$

is ________.

Answer (4)

Sol. $3x^2 - 4\left|x^2 - 1\right| + x - 1 = 0$

Let $x \in [-1, \ \ 1]$

$\Rightarrow \ 3x^2 - 4(-x^2 + 1) + x - 1 = 0$

$\Rightarrow \ 3x^2 + 4x^2 - 4 + x - 1 = 0$

$\Rightarrow \ 7x^2 + x - 5 = 0$

$\Rightarrow \ x = \dfrac{-1 \pm \sqrt{1 + 140}}{2}$

Both values acceptable

Let $x \in (-\infty, -1) \cup (1, \ \infty)$

$x^2 - 4(x^2 - 1) + x - 1 = 0$

$\Rightarrow \ x^2 - x - 3 = 0$

$\Rightarrow \ x = \dfrac{1 \pm \sqrt{1 + 12}}{2}$

Again both are acceptable

Hence total number of solution = 4

18. In a triangle *ABC*, let $AB = \sqrt{23}$, and *BC* = 3 and *CA* = 4. Then the value of

$$\frac{\cot A + \cot C}{\cot B}$$

is ______.

Answer (2)

Sol. With standard notations

Given : $c=\sqrt{23}$, a = 3, b = 4

Now $\dfrac{\cot A+\cot C}{\cot B}=\dfrac{\dfrac{\cos A}{\sin A}+\dfrac{\cos C}{\sin C}}{\dfrac{\cos B}{\sin B}}$

$$=\frac{\dfrac{b^2+c^2-a^2}{2bc.\sin A}+\dfrac{a^2+b^2-c^2}{2ab\sin C}}{\dfrac{c^2+a^2-b^2}{2ac\sin B}}$$

$$=\frac{\dfrac{b^2+c^2-a^2}{4\Delta}+\dfrac{a^2+b^2-c^2}{4\Delta}}{\dfrac{c^2+a^2-b^2}{4\Delta}}=\frac{2b^2}{a^2+c^2-b^2}=2$$

19. Let $\vec{u},\vec{v}$ and $\vec{w}$ be vectors in three-dimensional space, where $\vec{u}$ and $\vec{v}$ are unit vectors which are not perpendicular to each other and

$\vec{u}\cdot\vec{w}=1,\quad \vec{v}\cdot\vec{w}=1,\quad \vec{w}\cdot\vec{w}=4$

If the volume of the parallelopiped, whose adjacent sides are represented by the vectors $\vec{u},\vec{v}$ and $\vec{w}$, is $\sqrt{2}$, then the value of $|3\vec{u}+5\vec{v}|$ is ________.

Answer (7)

Sol. Given $\left[\vec{u}\ \ \vec{v}\ \ \vec{w}\right]=\sqrt{2}$

Also $\left[\vec{u}\ \ \vec{v}\ \ \vec{w}\right]^2=\begin{vmatrix}\vec{u}.\vec{u} & \vec{u}\cdot\vec{v} & \vec{u}\cdot\vec{w}\\ \vec{v}\cdot\vec{u} & \vec{v}\cdot\vec{v} & \vec{v}\cdot\vec{w}\\ \vec{w}\cdot\vec{u} & \vec{w}\cdot\vec{v} & \vec{w}\cdot\vec{w}\end{vmatrix}=2$

Let $\vec{u}.\vec{v}=k$ and substitute rest values, we get

$$\begin{vmatrix}1 & K & 1\\ K & 1 & 1\\ 1 & 1 & 4\end{vmatrix}=2$$

$\Rightarrow$ $4K^2 - 2K = 0$

$\Rightarrow$ $\vec{u}.\vec{v}=0$ or $\vec{u}.\vec{v}=\dfrac{1}{2}$

(rejected)

$\therefore$ $\vec{u}.\vec{v}=\dfrac{1}{2}$

$|3\vec{u}+5\vec{v}|^2=9+25+30\times\dfrac{1}{2}=49$

$\Rightarrow$ $|3\vec{u}+5\vec{v}|=7$

DATE : 03/10/2021

Time : 3 hrs. | Max. Marks: 180

Answers & Solutions

for

JEE (Advanced)-2021

PAPER - 2

PART-I : PHYSICS

SECTION - 1

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONE OR MORE THAN ONE** of these four option(s) is (are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme:

  Full Marks : +4 If only (all) the correct option(s) is(are) chosen;

  Partial Marks : +3 If all the four options are correct but ONLY three options are chosen;

  Partial Marks : +2 If three or more options are correct but ONLY two options are chosen, both of which are correct;

  Partial Marks : +1 If two or more options are correct but ONLY one option is chosen and it is a correct option;

  Zero Marks : 0 If unanswered;

  Negative Marks : -2 In all other cases.
- For example, in a question, if (A), (B) and (D) are the ONLY three options corresponding to correct answers, then

  choosing ONLY (A), (B) and (D) will get +4 marks;

  choosing ONLY (A) and (B) will get +2 marks;

  choosing ONLY (A) and (D) will get +2 marks;

  choosing ONLY (B) and (D) will get +2 marks;

  choosing ONLY (A) will get +1 mark;

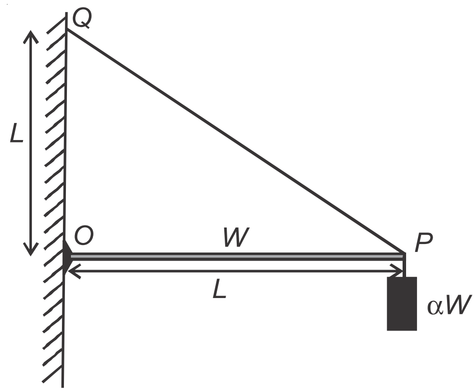
  choosing ONLY (B) will get +1 mark;

  choosing ONLY (D) will get +1 mark;

  choosing no option(s) (i.e. the question is unanswered) will get 0 marks and

  choosing any other option(s) will get -2 marks.

1. One end of a horizontal uniform beam of weight W and length L is hinged on a vertical wall at point O and its other end is supported by a light inextensible rope. The other end of the rope is fixed at point Q, at a height L above the hinge at point O. A block of weight αW is attached at the point P of the beam, as shown in the figure (not to scale). The rope can sustain a maximum tension of $(2\sqrt{2})W$. Which of the following statement(s) is(are) correct?



(A) The vertical compenent of reaction force at O does **not** depend on α

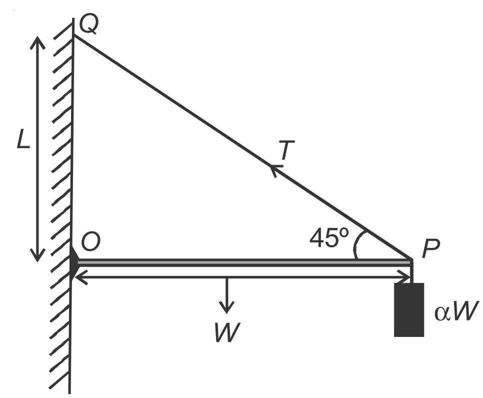
(B) The horizontal component of reaction force at O is equal to W for $\alpha = 0.5$

(C) The tension in the rope is $2W$ for $\alpha = 0.5$

(D) The rope breaks if $\alpha > 1.5$

Answer (A, B, D)

Sol. $W \times \frac{L}{2} + \alpha W \times L = T \times \frac{1}{\sqrt{2}} \times L$



$\Rightarrow \quad T = \sqrt{2}\left(\frac{1}{2} + \alpha\right)W$

$\therefore \quad T \times \frac{1}{\sqrt{2}} + F_V = W + \alpha W$

$\Rightarrow \quad \frac{W}{2} + \alpha W + F_V = W + \alpha W$

$\Rightarrow \quad F_V = \frac{W}{2}$

At $\alpha = \frac{1}{2}$, $\quad T = \sqrt{2}\left(\frac{1}{2} + \frac{1}{2}\right)W = \sqrt{2}W$

$F_H\left(at \;\; \alpha = \frac{1}{2}\right) = \sqrt{2}W \times \frac{1}{\sqrt{2}} = W$

at α = 1.5, $T = \sqrt{2} \times \left(\frac{1}{2} + \frac{3}{2}\right)W = 2\sqrt{2}W$

2. A source, approaching with speed u towards the open end of a stationary pipe of length L, is emitting a sound of frequency f_s. The farther end of the pipe is closed. The speed of sound in air is v and f_0 is the fundamental frequency of the pipe. For which of the following combination(s) of u and f_s, will the sound reching the pipe lead to a resonance?

(A) $u = 0.8v$ and $f_s = f_0$

(B) $u = 0.8v$ and $f_s = 2f_0$

(C) $u = 0.8v$ and $f_s = 0.5f_0$

(D) $u = 0.5v$ and $f_s = 1.5f_0$

Answer (A, D)

Sol. For resonance,

$$\frac{v}{v-u}\times f_s = (\text{odd})\times f_0$$

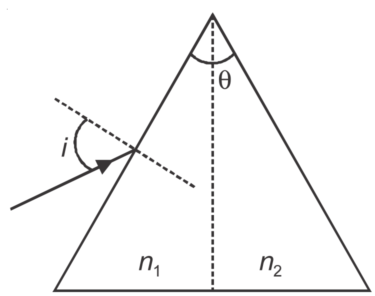
(A) $\frac{v}{v-0.8v}\times f_0 = 5f_0$

(B) $\frac{v}{v-0.8v}\times 2f_0 = 10f_0$

(C) $\frac{v}{v-0.8v}\times\left(\frac{f_0}{2}\right) = \frac{5}{2}f_0$

(D) $\frac{v}{v-0.5v}\times\left(\frac{3f_0}{2}\right) = 3f_0$

3. For a prism of prism angle $\theta = 60^\circ$, the refractive indices of the left half and the right half are, respectively, n_1 and n_2 ($n_2 \geq n_1$) as shown in the figure. The angle of incidence i is chosen such that the incident light rays will have minimum deviation if $n_1 = n_2 = n = 1.5$. For the case of unequal refractive indices, $n_1 = n$ and $n_2 = n + \Delta n$ (where $\Delta n << n$), the angle of emergence $e = i + \Delta e$. Which of the following statement(s) is(are) correct?



(A) The value of Δe (in radians) is greater than that of Δn

(B) Δe is proportional to Δn

(C) Δe lies between 2.0 and 3.0 milliradians, if $\Delta n = 2.8 \times 10^{-3}$

(D) Δe lies between 1.0 and 1.6 milliradians, if $\Delta n = 2.8 \times 10^{-3}$

Answer (B, C)

Sol. For $n_1 = n_2 = n = 1.5$,

$r = 30^\circ$

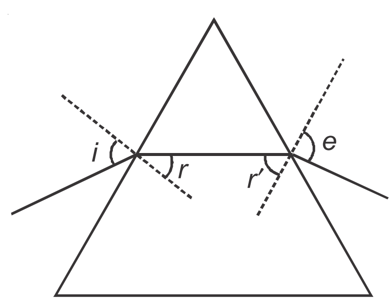
$\therefore$ $\sin i = 1.5 \times \sin(30^\circ) = \frac{3}{4}$

$\Rightarrow$ $\sin e = \frac{3}{4}$ for $n_1 = n_2$

Now, $r' = 30^\circ$ and $n_2 = n + \Delta n$

$$(n_2)\times\sin(r') = 1\times\sin e$$

$$\Rightarrow \quad \Delta n_2 \times \sin(30^\circ) = \cos e \times \Delta e$$



$$\Rightarrow \quad \Delta e = \frac{(\Delta n) \times \frac{1}{2}}{\sqrt{1 - \frac{9}{16}}} = \frac{2}{\sqrt{7}} \Delta n$$

$\Rightarrow \quad \Delta e < \Delta n$ and, $\Delta e \propto \Delta n$

at $\Delta n = 2.8 \times 10^{-3}$, $\Delta e = 2.12 \times 10^{-3}$ rad

4. A physical quantity $\vec{S}$ is defined as $\vec{S} = (\vec{E} \times \vec{B}) / \mu_0$, where $\vec{E}$ is electric field, $\vec{B}$ is magnetic field and μ_0 is the permeability of free space. The dimensions of $\vec{S}$ are the same as the dimensions of which of the following quantity(ies)?

(A) $\dfrac{\text{Energy}}{\text{Charge} \times \text{Current}}$

(B) $\dfrac{\text{Force}}{\text{Length} \times \text{Time}}$

(C) $\dfrac{\text{Energy}}{\text{Volume}}$

(D) $\dfrac{\text{Power}}{\text{Area}}$

Answer (B, D)

Sol. $\vec{S} = \dfrac{\vec{E} \times \vec{B}}{\mu_0}$

$\vec{S}$ is known as poynting vector and represents intensity of electromagnetic waves.

$$\left[\vec{S}\right] = [MT^{-3}] = \left[\frac{\text{Power}}{\text{Area}}\right] = \left[\frac{\text{Force}}{\text{Length} \times \text{Time}}\right]$$

5. A heavy nucleus N, at rest, undergoes fission $N \to P + Q$, where P and Q are two lighter nuclei. Let $\delta = M_N - M_P - M_Q$, where M_P, M_Q and M_N are the masses of P, Q and N, respectively. E_P and E_Q are the kinetic energies of P and Q, respectively. The speeds of P and Q are V_P and V_Q, respectively. If c is the speed of light, which of the following statement(s) is(are) correct?

(A) $E_P + E_Q = c^2\delta$

(B) $E_P = \left(\dfrac{M_P}{M_P + M_Q}\right) c^2 \delta$

(C) $\dfrac{V_P}{V_Q} = \dfrac{M_Q}{M_P}$

(D) The magnitude of momentum for P as well as Q is $c\sqrt{2\mu\delta}$, where $\mu = \dfrac{M_P M_Q}{(M_P + M_Q)}$

Answer (A, C, D)

Sol. $E_P + E_Q = \delta c^2$ (Q-value of nuclear reaction)

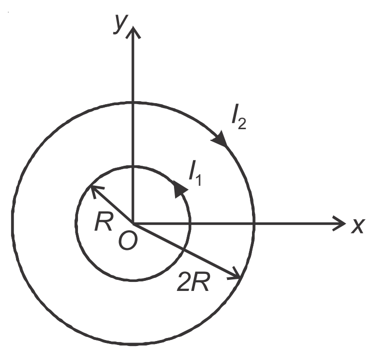
$$\sqrt{2M_P E_P} = \sqrt{2M_Q E_Q} \quad \text{or} \quad M_P V_P = M_Q V_Q$$

$$\Rightarrow \quad \frac{E_P}{E_Q} = \frac{M_Q}{M_P}$$

$$\Rightarrow \quad E_P = \frac{M_Q}{M_P + M_Q}\delta c^2$$

$\Rightarrow$ Momentum of P or Q = $\sqrt{\dfrac{2M_P M_Q}{M_P + M_Q}\delta c^2}$

6. Two concentric circular loops, one of radius R and the other of radius $2R$, lie in the xy-plane with the origin as their common center, as shwon in the figure. The smaller loop carries current I_1 in the anti-clockwise direction and the larger loop carries current I_2 in the clockwise direction, with $I_2 > 2I_1$. $\vec{B}(x, y)$ denotes the magnetic field at a point (x, y) in the xy-plane. Which of the following statement(s) is(are) correct?



(A) $\vec{B}(x, y)$ is perpendicular to the xy-plane at any point in the plane

(B) $\left|\vec{B}(x, y)\right|$ depends on x and y only through the radial distance $r = \sqrt{x^2 + y^2}$

(C) $\left|\vec{B}(x, y)\right|$ is non-zero at all points for $r < R$

(D) $\vec{B}(x, y)$ points normally outward from the xy-plane for all the points between the two loops

Answer (A, B)

Sol. Magnetic field due to a circular loop at any point in its plane will be perpendicular to the plane. Due to symmetry it will depend only on distance from centre. Field will be in opposite direction inside and outside the loop.

Field may be non-zero for $r < R$, as it is in opposite direction due to both the loops.

SECTION - 2

- This section contains **THREE (03)** question stems.
- There are **TWO (02)** questions corresponding to each question stem.
- The answer to each question is a **NUMERICAL VALUE.**
- For each question, enter the correct numerical value corresponding to the answer in the designated place using the mouse and the on-screen virtual numeric keypad.
- If the numerical value has more than two decimal places, **truncate/round-off** the value to **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme:

  Full Marks : +2 If ONLY the correct numerical value is entered at the designated place;

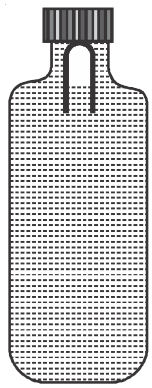
  Zero Marks : 0 In all other cases.

Question Stem for Question Nos. 7 and 8

Question Stem

A soft plastic bottle, filled with water of density 1 gm/cc, carries an inverted glass test-tube with some air (ideal gas) trapped as shown in the figure. The test-tube has a mass of 5 gm, and it is made of a thick glass of density 2.5 gm/cc. Initially the bottle is sealed at atmospheric pressure $p_0 = 10^5$ Pa so that the volume of the trapped air is $V_0 = 3.3$ cc. When the bottle is squeezed from outside at constant temperature, the pressure inside rises and the volume of the trapped air reduces. It is found that the test tube begins to sink at pressure $p_0 + \Delta p$ without changing its orientation. At this pressure, the volume of the trapped air is $V_0 - \Delta V$.

Let $\Delta V = X$ cc and $\Delta p = Y \times 10^3$ Pa.



7. The value of X is __________.

Answer (0.30)

8. The value of Y is __________.

Answer (10.00)

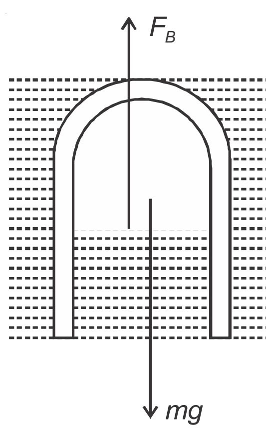
Solution of Q. Nos. 7 & 8

When bouyant force on (tube + air) system will become equal to weight of tube then tube will start sinking. (Here we can neglect weight of air as compared to weight of tube)

Now, Let volume of air in this case = V_{air}

$F_B = mg$

So, $\delta_w (V_{tube} + V_{air})\, g = mg$



$$\Rightarrow \quad 1\left(\frac{5}{2.5}\text{ cm}^3 + V_{air}\right) = 5$$

$\Rightarrow \quad 2 + V_{air} = 5$

$V_{air} = 3 \text{ cm}^3$

As initial volume of air = 3.3 cm^3

So, $\Delta V = 0.3$ cc

So, $X = 0.30$

As temperature of air is constant

So, PV = constant

$P_0\, 3.3 = P_f\, 3$, P_f is final pressure of air

$\Rightarrow \quad P_f = 1.1\, P_0 = P_0 + 0.1\, P_0$

So, $\Delta P = 10^4$ Pa

So, $Y = 10$

So, $X = 0.30$

$Y = 10.00$

Question Stem for Question Nos. 9 and 10

Question Stem

A pendulum consists of a bob of mass m = 0.1 kg and a massless inextensible string of length L = 1.0 m. It is suspended from a fixed point at height H = 0.9 m above a frictionless horizontal floor. Initially, the bob of the pendulum is lying on the floor at rest vertically below the point of suspension. A horizontal impulse P = 0.2 kg-m/s is imparted to the bob at some instant. After the bob slides for some distance, the string becomes taut and the bob lifts off the floor. The magnitude of the angular momentum of the pendulum about the point of suspension just before the bob lifts off is J kg-m^2/s. The kinetic energy of the pendulum just after the lift-off is K Joules.

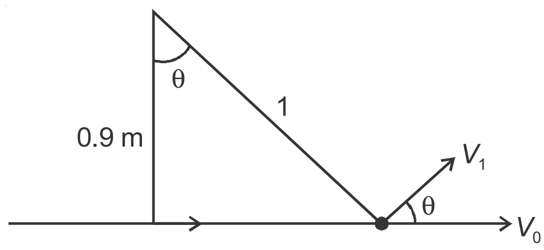
9. The value of J is __________.

Answer (0.18)

10. The value of K is __________.

Answer (0.16)

Solution of Q. Nos. 9 & 10



$$V_0 = \frac{0.2}{0.1} = 2 \text{ m/s}$$

9. $L = P \times H$

$= 0.2 \times 0.9$

$= 0.18$ kg m^2/s

10. $V_1 = V_0 \cos\theta = 2 \times \left(\frac{0.9}{1}\right)$

$\therefore \quad K = \frac{1}{2} \times (0.1) \times (2 \times 0.9)^2 = 0.162$ Joules

Question Stem for Question Nos. 11 and 12

Question Stem

In a circuit, a metal filament lamp is connected in series with a capacitor of capacitance C μF across a 200 V, 50 Hz supply. The power consumed by the lamp is 500 W while the voltage drop across it is 100 V. Assume that there is no inductive load in the circuit. Take *rms* values of the voltages. The magnitude of the phase-angle (in degrees) between the current and the supply voltage is φ. Assume, $\pi\sqrt{3} \approx 5$.

11. The value of C is ___.

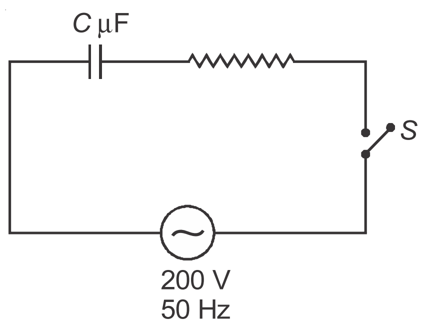
Answer (100)

Sol.

12. The value of φ is ___.

Answer (60)

Solution of Q. Nos. 11 & 12



$$P = \frac{V^2}{R} \Rightarrow 500 = \frac{100^2}{R}$$

$\Rightarrow$ $R = 20\ \Omega$

Now across resistance

$500 = I \times 100$

$\Rightarrow$ $I_{rms} = 5$ A

$V_{rms} = 200$ V, $V_{rms/real} = 100$ V

$$\cos\phi = \frac{100}{200} = \frac{1}{2} \Rightarrow \phi = 60^\circ$$

$$\tan\phi = \frac{X_C}{R} = \frac{1}{\omega RC}$$

$$\sqrt{3} = \frac{1}{100\pi(20)C}$$

$$\Rightarrow \quad C = \frac{1}{20\pi\sqrt{3}\times 100} = 10^{-4}\ \text{F} = 100\ \mu\text{F}$$

SECTION - 3

- This section contains **TWO (02) paragraphs.** Based on each paragraph, there are **TWO (02)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme:

  Full Marks : Full Marks : +3 If ONLY the correct option is chosen;

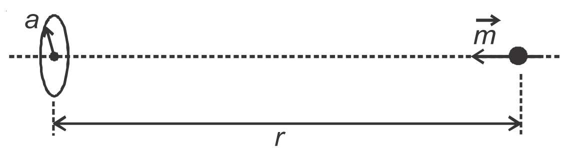
  Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered);

  Negative Marks : -1 In all other cases.

Paragraph

A special metal *S* conducts electricity without any resistance. A closed wire loop, made of *S*, does not allow any change in flux through itself by inducing a suitable current to generate a compensating flux. The induced current in the loop cannot decay due to its zero resistance. This current gives rise to a magnetic moment which in turn repels the source of magnetic field or flux. Consider such a loop, of radius *a*, with its center at the origin. A magnetic dipole of moment *m* is brought along the axis of this loop from infinity to a point at distance *r* (>> *a*) from the center of the loop with its north pole always facing the loop, as shown in the figure below.

The magnitude of magnetic field of a dipole m, at a point on its axis at distance r, is $\frac{\mu_0 m}{2\pi r^3}$, where μ_0 is the permeability of free space. The magnitude of the force between two magnetic dipoles with moments, m_1 and m_2, separated by a distance r on the common axis, with their north poles facing each other, is $\frac{km_1m_2}{r^4}$, where k is a constant of appropriate dimensions. The direction of this force is along the line joining the two dipoles.



13. When the dipole m is placed at a distance r from the center of the loop (as shown in the figure), the current induced in the loop will be proportional to

(A) m/r^3

(B) m^2/r^2

(C) m/r^2

(D) m^2/r

Answer (A)

Sol. Magnetic flux due to dipole through ring = $\frac{\mu_0}{2\pi}\times\frac{m}{r^3}\times\pi a^2$

for net magnetic flux through the loop to be zero.

Magnetic flux due to dipole = Magnetic flux due to induced current

$\Rightarrow \frac{\mu_0}{2\pi}\times\pi a^2\times\frac{m}{r^3} = I\times\pi a^2\times\frac{k}{a}$, where k is proportionality constant.

$\Rightarrow I \propto \frac{m}{r^3}$

14. The work done in bringing the dipole from infinity to a distance r from the center of the loop by the given process is proportional to

(A) m/r^5

(B) m^2/r^5

(C) m^2/r^6

(D) m^2/r^7

Answer (C)

Sol. $F = \frac{km_1m_2}{r^4} = k\left(I\pi a^2\right)\left(\frac{m}{r^4}\right)$

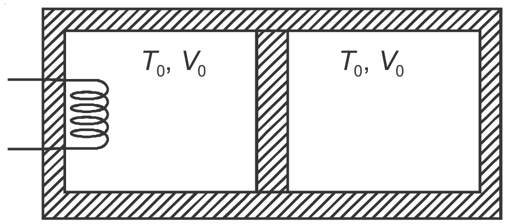
$F = C\frac{m^2}{r^7}$ where C is a constant obtained by substituting the value of I from Q.13

$|W| = \int_{\infty}^{r} F\,dr = Cm^2\int_{\infty}^{r}\frac{dr}{r^7} = \frac{C'm^2}{r^6}$ where C' is a constant

$|W| \propto \frac{m^2}{r^6}$

Paragraph

A thermally insulating cylinder has a thermally insulating and frictionless movable partition in the middle, as shown in the figure below. On each side of the partition, there is one mole of an ideal gas, with specific heat at constant volume, $C_V = 2R$. Here, R is the gas constant. Initially, each side has a volume V_0 and temperature T_0. The left side has an electric heater, which is turned on at very low power to transfer heat Q to the gas on the left side. As a result the partition moves slowly towards the right reducing the right side volume to $V_0/2$. Consequently, the gas temperatures on the left and the right sides become T_L and T_R, respectively. Ignore the changes in the temperatures of the cylinder, heater and the partition.



15. The value of $\frac{T_R}{T_0}$ is

(A) $\sqrt{2}$
(B) $\sqrt{3}$
(C) 2
(D) 3

Answer (A)

Sol. $PV^\gamma = C$

$\Rightarrow TV^{\gamma-1} = C$

$$\Rightarrow T_0 V_0^{\gamma-1} = T_R \left(\frac{V_0}{2}\right)^{\gamma-1}$$

$$C_V = \frac{R}{\gamma-1}$$

$$\Rightarrow 2R = \frac{R}{\gamma-1}$$

$$\Rightarrow \gamma-1 = \frac{1}{2}$$

$$\Rightarrow \gamma = \frac{3}{2}$$

$$\Rightarrow \frac{T_R}{T_0} = 2^{\gamma-1} = \sqrt{2}$$

16. The value of $\frac{Q}{RT_0}$ is

(A) $4(2\sqrt{2}+1)$
(B) $4(2\sqrt{2}-1)$
(C) $(5\sqrt{2}+1)$
(D) $(5\sqrt{2}-1)$

Answer (B)

Sol. $Q = \Delta U_1 + \Delta U_2$

$\Delta U_1 = C_V \Delta T_1 = 2R(T_L - T_0)$

$\Delta U_2 = C_V \Delta T_2 = 2R(T_R - T_0)$

$T_L = 3\sqrt{2}T_0$, $T_R = \sqrt{2}T_0$

$$Q = 2R\left[3\sqrt{2} - 1\right]T_0 + 2R(\sqrt{2} - 1)T_0$$

$$Q = 4RT_0[2\sqrt{2} - 1]$$

$$\Rightarrow \frac{Q}{RT_0} = 4[2\sqrt{2} - 1]$$

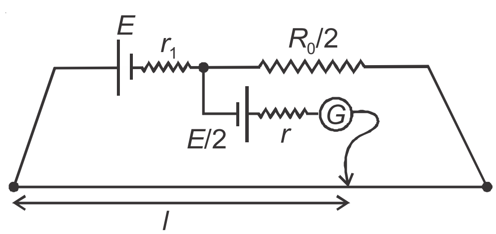
SECTION - 4

- This section contains THREE (03) questions.
- The answer to each question is a NON-NEGATIVE INTEGER.
- For each question, enter the correct integer corresponding to the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme:

  Full Marks : +4 If ONLY the correct integer is entered;

  Zero Marks : 0 In all other cases.

17. In order to measure the internal resistance r_1 of a cell of emf E, a meter bridge of wire resistance $R_0 = 50\ \Omega$, a resistance $R_0/2$, another cell of emf $E/2$ (internal resistance r) and a galvanometer G are used in a circuit, as shown in the figure. If the null point is found at $l = 72$ cm, then the value of r_1 = ___ Ω.



Answer (3)

Sol. Current will flow in main circuit

$$I = \frac{E}{r_1 + \frac{3R_0}{2}}$$

$$+E - IR_0 \times 0.72 - Ir_1 - \frac{E}{2} = 0$$

$$\frac{E}{2} = \frac{2E}{2r_1 + 3R_0} \times \left[0.72R_0 + r_1\right]$$

$$2r_1 + 3R_0 = 4\left[0.72R_0 + r_1\right]$$

$0.12R_0 = 2r_1$

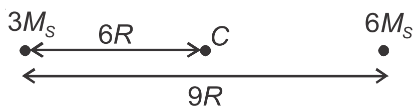
$r_1 = 3\Omega$

18. The distance between two stars of masses $3M_S$ and $6M_S$ is $9R$. Here R is the mean distance between the centers of the Earth and the Sun, and M_S is the mass of the Sun. The two stars orbit around their common center of mass in circular orbits with period nT, where T is the period of Earth's revolution around the Sun.

The value of n is ___.

Answer (9)

Sol.



Centre of mass of system lies at $6R$ from lighter mass.

$$\left[3M_S\omega^2 \times 6R\right] = \frac{G(18M_S^2)}{81R^2}$$

$$\omega^2 R = \frac{GM_S}{81R^2}$$

$$T' = \sqrt{\frac{81R^3}{GM_S}}$$

$$T' = 9T$$

$n = 09$

19. In a photoemission experiment, the maximum kinetic energies of photoelectrons from metals P, Q and R are E_P, E_Q and E_R, respectively, and they are related by $E_P = 2E_Q = 2E_R$. In this experiment, the same source of monochromatic light is used for metals P and Q while a different source of monochromatic light is used for the metal R. The work functions for metals P, Q and R are 4.0 eV, 4.5 eV and 5.5 eV, respectively. The energy of the incident photon used for metal R, in eV, is ___.

Answer (6)

Sol. $$\frac{hc}{\lambda_1} = \varphi_P + E_P$$

$$\frac{hc}{\lambda_1} = \varphi_Q + E_Q$$

$E_P = 2E_Q$

$E_P - E_Q = 0.5$

$\Rightarrow$ $E_P = 1.0$ eV, $E_Q = 0.5$ eV

$E_R = 0.5$ eV

Energy of incident photon on $R = \varphi_R + E_R = 6$ eV

PART-II : CHEMISTRY

SECTION - 1

- This section contains **EIGHT (08)** questions.
- Each question has **FOUR** options (A), (B), (C) & (D). **ONE OR MORE THAN ONE** of these four option(s) is(are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +4 If only (all) the correct option(s) is(are) chosen;

Partial Marks : +3 If all the four options are correct but ONLY three options are chosen;

Partial Marks : +2 If three or more options are correct but ONLY two options are chosen, and both of which are correct;

Partial Marks : +1 If two or more options are correct but ONLY one option is chosen and it is a correct option;

Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered);

Negative Marks : –2 In all other cases.

- For example : in a question, if (A), (B) and (D) are the ONLY three options corresponding to correct answers, then

choosing ONLY (A), (B) and (D) will get +4 marks;

choosing ONLY (A) and (B) will get +2 marks;

choosing ONLY (A) and (D) will get +2 marks;

choosing ONLY (B) and (D) will get +2 marks;

choosing ONLY (A) will get +1 mark;

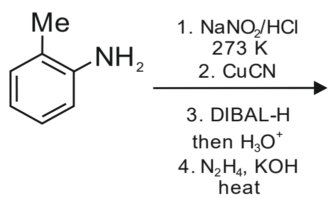
choosing ONLY (B) will get +1 mark;

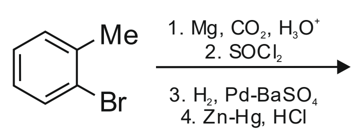
choosing ONLY (D) will get +1 mark;

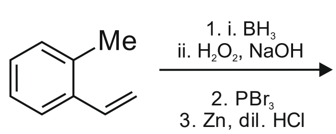
choosing no option (i.e., the question is unanswered) will get 0 marks; and

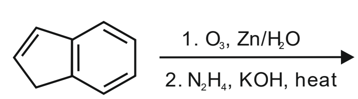
choosing any other combination of options will get –2 mark.

1. The reaction sequence(s) that would lead to *o*-xylene as the major product is(are)

(A) 

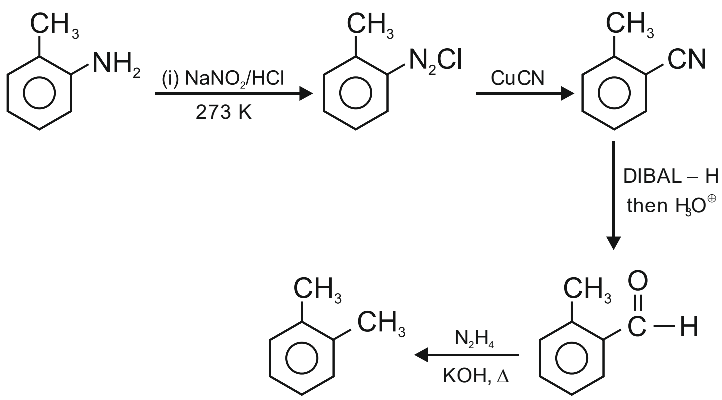
(B) 

(C) 

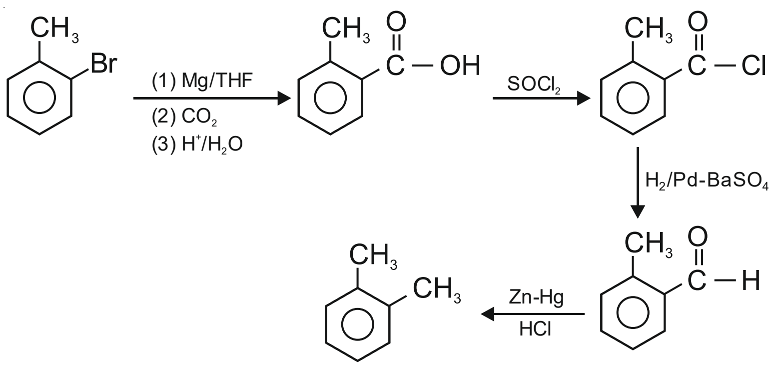
(D) 

Answer (A, B)

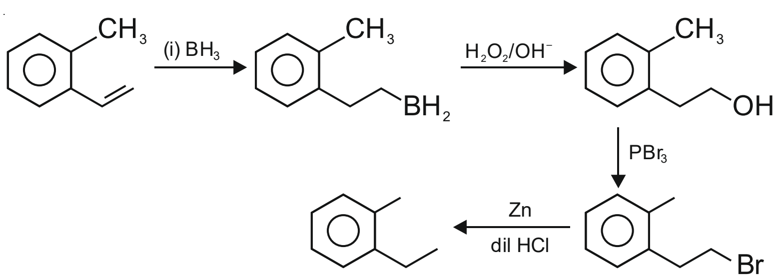
Sol. (A)



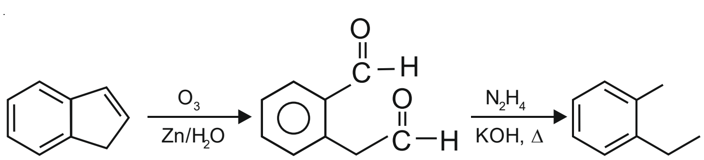
(B)



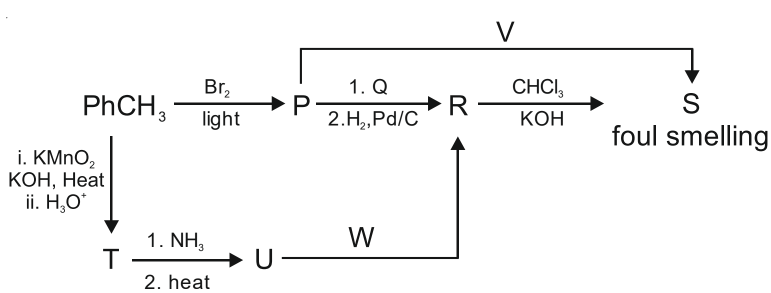
(C)



(D)



2. Correct option(s) for the following sequence of reactions is(are)



(A) Q = KNO_2, W = $LiAlH_4$

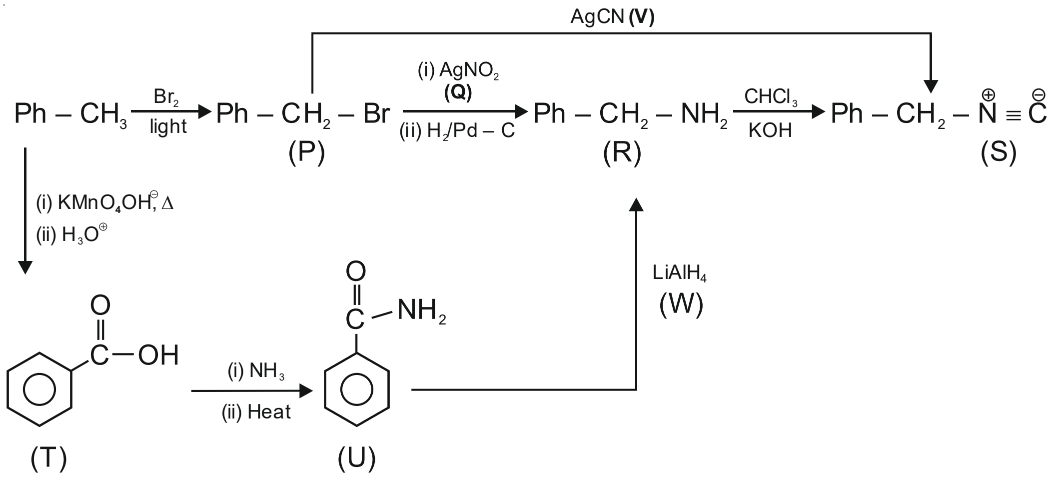
(B) R = benzenamine, V = KCN

(C) Q = $AgNO_2$, R = phenylmethanamine

(D) W = $LiAlH_4$, V = AgCN

Answer (C, D)

Sol.



∴ Correct option are

$Q = AgNO_2$, R = phenylmethanamine

$W = LiAlH_4$, $V = AgCN$

3. For the following reaction

$$2X + Y \xrightarrow{k} P$$

the rate of reaction is $\frac{d[P]}{dt} = k[X]$. Two moles of X are mixed with one mole of Y to make 1.0 L of solution. At 50 s, 0.5 mole of Y is left in the reaction mixture. The correct statement(s) about the reaction is(are)

(Use: ln 2 = 0.693)

(A) The rate constant, k, of the reaction is $13.86 \times 10^{-4}\ s^{-1}$.

(B) Half-life of X is 50 s.

(C) At 50 s, $-\frac{d[X]}{dt} = 13.86 \times 10^{-3}$ mol $L^{-1}\ s^{-1}$.

(D) At 100 s, $-\frac{d[Y]}{dt} = 3.46 \times 10^{-3}$ mol $L^{-1}\ s^{-1}$.

Answer (B,C,D)

Sol. rate $= \frac{d[P]}{dt} = k[X]$

2X	+ Y	→ P
2 mole	1 mole	
1 mole	0.5 mole	0.5 mole

$$-\frac{d[X]}{dt} = k_1[X] = 2k[X] \Rightarrow 2k = k_1$$

$$-\frac{d[Y]}{dt} = k_2[X] = k[X] \Rightarrow k_2 = k$$

$$2k = \frac{1}{50}\ln 2$$

$$k = \frac{1}{100}\ln 2 = \frac{0.693}{100} = 6.93 \times 10^{-3}\ s^{-1}$$

$$(t_{1/2})_x = \frac{\ln 2}{k_1} = \frac{\ln 2 \times 100}{2 \times 0.693} = 50\ sec$$

At 50 sec

$$-\frac{d[X]}{dt} = 2k[X] = 2 \times \frac{0.693}{100} \times 1$$

$= 13.86 \times 10^{-3}$ mol L^{-1} s^{-1}

At 100 sec

$$-\frac{d[Y]}{dt} = k_2[X] = k[X] = \frac{0.693}{100} \times \frac{1}{2}$$

($\because$ Concentration of X after 2 half lives = $\frac{1}{2}$ M)

$= 3.46 \times 10^{-3}$ mol L^{-1} s^{-1}

4. Some standard electrode potentials at 298 K are given below:

Pb^{2+}/Pb	–0.13 V
Ni^{2+}/Ni	–0.24 V
Cd^{2+}/Cd	–0.40 V
Fe^{2+}/Fe	–0.44 V

To a solution containing 0.001 M of X^{2+} and 0.1 M of Y^{2+}, the metal rods X and Y are inserted (at 298 K) and connected by a conducting wire. This resulted in dissolution of X. The correct combination(s) of X and Y, respectively, is(are)

(Given: Gas constant, R = 8.314 J K^{-1} mol^{-1}, Faraday constant, F = 96500 C mol^{-1})

(A) Cd and Ni (B) Cd and Fe

(C) Ni and Pb (D) Ni and Fe

Answer (A,B,C)

Sol. $X + Y^{2+} \rightarrow X^{2+} + Y$

$$E = E^\circ - \frac{0.06}{2}\log_{10}\left(\frac{10^{-3}}{10^{-1}}\right)$$

$E = E^\circ + 0.06$

(A) $E^\circ = -(-.4) + (-.24) = .16 > 0$

(B) $E^\circ = -(-.4) + (-.44) = -.04 < 0$ and $E_{cell} = -0.04 + 0.06 = +0.02 > 0$

(C) $E^\circ = -(-.24) + (-.13) = .11 > 0$

(D) $E^\circ = -(-.24) + (-.44) = -.2 < 0$

$\therefore$ $E_{cell} = -0.2 + 0.06 = -0.14 < 0$

$\therefore$ If $E_{cell} > 0$ then the cell construction is possible.

5. The pair(s) of complexes wherein both exhibit tetrahedral geometry is(are)

(Note: py = pyridine

Given: Atomic numbers of Fe, Co, Ni and Cu are 26, 27, 28 and 29, respectively)

(A) $[FeCl_4]^-$ and $[Fe(CO)_4]^{2-}$

(B) $[Co(CO)_4]^-$ and $[CoCl_4]^{2-}$

(C) $[Ni(CO)_4]$ and $[Ni(CN)_4]^{2-}$

(D) $[Cu(py)_4]^+$ and $[Cu(CN)_4]^{3-}$

Answer (A,B,D)

Sol. $[FeCl_4]^- \rightarrow Fe^{3+}$, $3d^5$ (weak field ligand) = sp^3

$[Fe(CO)_4]^{-2} \rightarrow Fe^{2-}$, $3d^{10} \rightarrow sp^3$

$[Co(CO)_4]^- \rightarrow Co^-$, $3d^{10} \rightarrow sp^3$

$[CoCl_4]^{2-} \rightarrow Co^{2+}$, $3d^7$ (weak field ligand) $\rightarrow sp^3$

$[Ni(CO)_4] \rightarrow Ni$, $3d^{10} \rightarrow sp^3$

$[Ni(CN)_4]^{2-} \rightarrow Ni^{2+}$, $3d^8$ (strong field ligand) $\rightarrow dsp^2$

$[Cu(py)_4]^+ \rightarrow Cu^+$, $3d^{10} \rightarrow sp^3$

$[Cu(CN)_4]^{3-} \rightarrow Cu^+$, $3d^{10} \rightarrow sp^3$

In $3d^{10}$ electronic configuration only sp^3 hybridisation and tetrahedral geometry is possible.

6. The correct statement(s) related to oxoacids of phosphorous is(are)

(A) Upon heating, H_3PO_3 undergoes disproportionation reaction to produce H_3PO_4 and PH_3.

(B) While H_3PO_3 can act as reducing agent, H_3PO_4 cannot.

(C) H_3PO_3 is a monobasic acid.

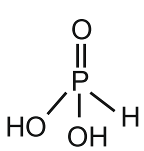
(D) The H atom of P-H bond in H_3PO_3 is not ionizable in water.

Answer (A,B,D)

Sol. $4H_3PO_3 \xrightarrow{\Delta} PH_3 + 3H_3PO_4$

In H_3PO_4, phosphorous is present in highest oxidation state, i.e., +5. So H_3PO_4 cannot acts as reducing agent.

Structure of H_3PO_3,



It is a dibasic acid.

H atom present in P–H bond is not ionizable.

These P-H bonds are not ionisable to give H^+ and do not play any role in basicity. Only those H atoms which are attached with oxygen in P-OH form are ionisable and cause the basicity. Thus, H_3PO_3 and H_3PO_4 are dibasic and tribasic, respectively as the structure of H_3PO_3 has two P–OH bonds and H_3PO_4 three.

SECTION - 2

- This section contains **THREE (03)** question stems.
- There are TWO (02) questions corresponding to each question stem.
- The answer to each question is a NUMERICAL VALUE.
- For each question, enter the correct numerical value corresponding to the answer in the designated place using the mouse and the on-screen virtual numeric keypad.
- If the numerical value has more than two decimal places, **truncate/round-off** the value to **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks	:	+2	If ONLY the correct numerical value is entered at the designated place.
Zero Marks	:	0	In all other cases.

Question Stem for Question Nos. 7 and 8

Question Stem

At 298 K, the limiting molar conductivity of a weak monobasic acid is 4×10^2 S cm^2 mol^{-1}. At 298 K, for an aqueous solution of the acid the degree of dissociation is α and the molar conductivity is $y \times 10^2$ S cm^2 mol^{-1}. At 298 K, upon 20 times dilution with water, the molar conductivity of the solution becomes $3y \times 10^2$ S cm^2 mol^{-1}.

7. The value of α is _______.

Answer (0.215)

8. The value of y is _______.

Answer (0.86)

Sol. Solution of Question Nos. 7 and 8

Molar conductivity of HX at infinite dilution

$\Lambda_m^\infty = 4 \times 10^2$ S cm^2 mol^{-1}

Molar conductivity of HX at conc. $c_1 = y \times 10^2$ S cm^2 mol^{-1}

$$\alpha_1 = \frac{\Lambda_m^{c_1}}{\Lambda_m^\infty} = \frac{y \times 10^2}{4 \times 10^2} = \frac{y}{4}$$

On 20 times dilution of the solution of HX

$$\alpha_2 = \frac{\Lambda_m^{c_2}}{\Lambda_m^\infty} = \frac{3y \times 10^2}{4 \times 10^2} = \frac{3y}{4} \qquad \left[c_2 = \frac{c_1}{20}\right]$$

$$\frac{\alpha_1}{\alpha_2} = \frac{1}{3} \qquad \Rightarrow \alpha_2 = 3\alpha_1$$

$$\begin{array}{ccccc} HX & \rightleftharpoons & H^+ & + & X^- \\ c_1(1-\alpha_1) & & c_1\alpha_1 & & c_1\alpha_1 \end{array}$$

$$K_a = \frac{c_1\alpha_1^2}{1-\alpha_1} = \frac{c_2\alpha_2^2}{1-\alpha_2} = \frac{c_1(3\alpha_1)^2}{20(1-3\alpha_1)}$$

$$\frac{1}{1-\alpha_1} = \frac{9}{20(1-3\alpha_1)}$$

$20 - 60\alpha_1 = 9 - 9\alpha_1 \quad \Rightarrow \quad \alpha_1 = \frac{11}{51} = 0.215$

$y = 4\alpha_1 = 0.86$

Question Stem for Question Nos. 9 and 10

Question Stem

Reaction of x g of Sn with HCl quantitatively produced a salt. Entire amount of the salt reacted with y g of nitrobenzene in the presence of required amount of HCl to produce 1.29 g of an organic salt (quantitatively).

(Use Molar masses (in g mol^{-1}) of H, C, N, O, Cl and Sn as 1, 12, 14, 16, 35 and 119, respectively).

9. The value of x is ________.

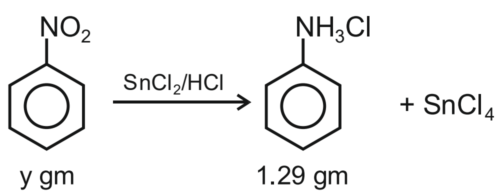
Answer (3.57)

10. The value of y is ________.

Answer (1.23)

Sol. Solution of Question Nos. 9 and 10

$Sn + HCl \rightarrow SnCl_2$



$\Rightarrow$ Moles of ammonium salt = $\frac{1.29}{129}$ = 0.01

$\Rightarrow$ Moles of nitrobenzene = 0.01

$\Rightarrow$ y = 0.01 × Molar mass of nitrobenzene

= 0.01 × 123

y = 1.23

Also

No. of eq. of nitrobenzene = No. of eq. of $SnCl_2$

$6 \times (0.01) = 2 \times n_{SnCl_2}$

$n_{SnCl_2} = 0.03$

$\Rightarrow$ $n_{Sn} = 0.03$

$w_{Sn} = 0.03 \times 119$

x = 3.57

Question Stem for Question Nos. 11 and 12

Question Stem

A sample (5.6 g) containing iron is completely dissolved in cold dilute HCl to prepare a 250 mL of solution. Titration of 25.0 mL of this solution requires 12.5 mL of 0.03 M $KMnO_4$ solution to reach the end point. Number of moles of Fe^{2+} present in 250 mL solution is x × 10^{-2} (consider complete dissolution of $FeCl_2$). The amount of iron present in the sample is y% by weight.

(Assume: $KMnO_4$ reacts only with Fe^{2+} in the solution

Use: Molar mass of iron as 56 g mol^{-1})

11. The value of x is ________.

Answer (1.875)

12. The value of y is ________.

Answer (18.75)

Sol. Solution of Question Nos. 11 and 12

$8H^+ + 5Fe^{2+} + MnO_4^- \rightarrow 5Fe^{3+} + Mn^{2+} + 4H_2O$

For 25 ml,

meq of Fe^{2+} = meq of MnO_4^-

= 12.5 × 0.03 × 5

For 250 ml,

mmoles of $Fe^{2+} = \dfrac{12.5 \times 0.03 \times 5 \times 250}{25}$

moles of $Fe^{2+} = \dfrac{18.75}{1000}$ mol

= 18.75 × 10^{-3} mol

= 1.875 × 10^{-2} mol

x = 1.875

Weight of Fe^{2+} = 1.875 × 10^{-2} × 56 = 1.05 g

% purity of $Fe^{2+} = \dfrac{1.05}{5.6} \times 100$ = 18.75%

y = 18.75%

SECTION - 3

- This section contains **TWO (02) paragraphs.** Based on each paragraph, there are **TWO (02)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks	:	+3	If ONLY the correct option is chosen;
Zero Marks	:	0	If none of the options is chosen (i.e. the question is unanswered);
Negative Marks	:	–1	In all other cases.

Paragraph

The amount of energy required to break a bond is same as the amount of energy released when the same bond is formed. In gaseous state, the energy required for homolytic cleavage of a bond is called Bond Dissociation Energy (BDE) or Bond Strength. BDE is affected by *s*-character of the bond and the stability of the radicals formed. Shorter bonds are typically stronger bonds. BDEs for some bonds are given below:

$H_3C-H(g) \longrightarrow H_3\dot{C}(g) + \dot{H}(g)\ \Delta H° = 105\ kcal\ mol^{-1}$

$Cl-Cl(g) \longrightarrow \dot{Cl}(g) + \dot{Cl}(g)\ \Delta H° = 58\ kcal\ mol^{-1}$

$H_3C-Cl(g) \longrightarrow H_3\dot{C}(g) + \dot{Cl}(g)\ \Delta H° = 85\ kcal\ mol^{-1}$

$H-Cl(g) \longrightarrow \dot{H}(g) + \dot{Cl}(g)\ \Delta H° = 103\ kcal\ mol^{-1}$

13. Correct match of the **C-H** bonds (shown in bold) in Column **J** with their BDE in Column **K** is

Column J	**Column K**
Molecule	**BDE (kcal mol^{-1})**
(P) **H-C**H$(CH_3)_2$	(i) 132
(Q) **H-C**H_2Ph	(ii) 110
(R) **H-C**H=CH_2	(iii) 95
(S) **H-C**≡CH	(iv) 88

(A) P – iii, Q – iv, R – ii, S – i

(B) P – i, Q – ii, R – iii, S – iv

(C) P – iii, Q – ii, R – i, S – iv

(D) P – ii, Q – i, R – iv, S – iii

Answer (A)

Sol. $H-CH(CH_3)_2 \longrightarrow H\dot{C}(CH_3)-CH_3$

$H-CH_2Ph \longrightarrow {}^{\bullet}CH_2-Ph$ (Benzyl radical)

$H-CH=CH_2 \rightarrow {}^{\bullet}CH=CH_2$

$H-C\equiv CH \rightarrow {}^{\bullet}C\equiv CH$

Order of stability of free radical

Q > P > R > S

Stability of free radical $\propto \dfrac{1}{\text{Bond energy}}$

$\therefore$ Order of bond energy :

S > R > P > Q

14. For the following reaction

$$CH_4(g) + Cl_2(g) \xrightarrow{\text{light}} CH_3Cl(g) + HCl(g)$$

the correct statement is

(A) Initiation step is exothermic with $\Delta H° = -58$ kcal mol^{-1}

(B) Propagation step involving $^{\bullet}CH_3$ formation is exothermic with $\Delta H° = -2$ kcal mol^{-1}

(C) Propagation step involving CH_3Cl formation is endothermic with $\Delta H° = +27$ kcal mol^{-1}

(D) The reaction is exothermic with $\Delta H° = -25$ kcal mol^{-1}

Answer (D)

Sol. (1) $Cl_2 \rightarrow 2Cl^{\bullet}$ (Initiation step) ΔH = 58 kcal/mol

$$\left.\begin{array}{l}(2)\ CH_4 + Cl^{\bullet} \rightarrow {}^{\bullet}CH_3 + HCl \\ (3)\ {}^{\bullet}CH_3 + Cl_2 \rightarrow CH_3Cl + Cl^{\bullet}\end{array}\right] \text{Propagation step}$$

Step (1) $\rightarrow$ Endothermic (bond breaking)

Step (2) $\rightarrow \Delta H = 105 - 103$

$= 2$ kcal/mol (Endothermic)

Step (3) $\rightarrow \Delta H = 58 - 85$

$= -27$ kcal/mol (Exothermic)

For complete reaction

$$CH_4(g) + Cl_2(g) \xrightarrow{\text{light}} CH_3Cl(g) + HCl(g)$$

$\Delta H = 58 + 105 - 85 - 103$

$= -25$ kcal/mol

Paragraph

The reaction of $K_3[Fe(CN)_6]$ with freshly prepared $FeSO_4$ solution produces a dark blue precipitate called Turnbull's blue. Reaction of $K_4[Fe(CN)_6]$ with the $FeSO_4$ solution in complete absence of air produces a white precipitate X, which turns blue in air. Mixing the $FeSO_4$ solution with $NaNO_3$, followed by a slow addition of concentrated H_2SO_4 through the side of the test tube produces a brown ring.

15. Precipitate X is

(A) $Fe_4[Fe(CN)_6]_3$

(B) $Fe[Fe(CN)_6]$

(C) $K_2Fe[Fe(CN)_6]$

(D) $KFe[Fe(CN)_6]$

Answer (C)

16. Among the following, the brown ring is due to the formation of

(A) $[Fe(NO)_2(SO_4)_2]^{2-}$

(B) $[Fe(NO)_2(H_2O)_4]^{3+}$

(C) $[Fe(NO)_4(SO_4)_2]$

(D) $[Fe(NO)(H_2O)_5]^{2+}$

Answer (D)

Sol. Solution of Question Nos. 15 and 16

$$Fe^{2+} + K_3[Fe(CN)_6] \rightarrow \underset{\text{Turnbull's blue ppt.}}{Fe_3[Fe(CN)_6]_2} \downarrow$$

$$Fe^{2+} + K_4[Fe(CN)_6] \xrightarrow[\text{of air}]{\text{in absence}} \underset{\text{White ppt. (X)}}{K_2Fe[Fe(CN)_6]} \downarrow$$

In air Fe^{2+} gets oxidised to Fe^{3+}

$$Fe^{3+} + [Fe(CN)_6]^{4-} \longrightarrow \underset{\text{Prussian blue}}{Fe_4[Fe(CN)_6]_3} \downarrow$$

$$2NO_3^- + 4H_2SO_4 + 6Fe^{2+} \rightarrow 6Fe^{3+} + 2NO\uparrow + 4SO_4^{2-} + 4H_2O$$

$$[Fe(H_2O)_6]^{2+} + NO \longrightarrow \underset{\substack{\text{Compound responsible}\\ \text{for brown ring}}}{[Fe(H_2O)_5NO]^{2+}} + H_2O$$

$\therefore$ X = $K_2Fe[Fe(CN)_6]$

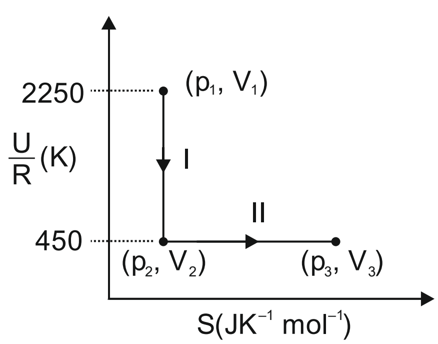
Brown ring is due to $[Fe(H_2O)_5NO]^{2+}$

SECTION - 4

- This section contains THREE (03) questions.
- The answer to each question is a NON-NEGATIVE INTEGER.
- For each question, enter the correct integer corresponding to the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks	:	+4	If ONLY the correct integer is entered;
Zero Marks	:	0	In all other cases.

17. One mole of an ideal gas at 900 K, undergoes two reversible processes, I followed by II, as shown below. If the work done by the gas in the two processes are same, the value of $\ln\frac{V_3}{V_2}$ is ____.



(U: internal energy, S: entropy, p: pressure, V: volume, R: gas constant)

(Given: molar heat capacity at constant volume, $C_{V,m}$ of the gas is $\frac{5}{2}R$)

Answer (10)

Sol. Process I is adiabatic reversible

Process II is reversible isothermal process

Process I - (Adiabatic Reversible)

$$\frac{\Delta U}{R} = 450 - 2250$$

$\Delta U = -1800R$

$W_I = \Delta U = -1800R$

Process II - (Reversible Isothermal Process)

$T_1 = 900$ K

Calculation of T_2 after reversible adiabatic process

$-1800R = nC_v(T_2 - T_1)$

$$-1800R = 1 \times \frac{5}{2}R(T_2 - 900)$$

$T_2 = 180$ K

$$W_{II} = -nRT_2 \ln\frac{v_3}{v_2} = W_I$$

$$\Rightarrow \quad -1 \times R \times 180 \ln\frac{v_3}{v_2} = -1800R$$

$$\ln\frac{v_3}{v_2} = 10$$

18. Consider a helium (He) atom that absorbs a photon of wavelength 330 nm. The change in the velocity (in cm s^{-1}) of He atom after the photon absorption is_____.

(Assume: Momentum is conserved when photon is absorbed.

Use: Planck constant = 6.6×10^{-34} J s, Avogadro number = 6×10^{23} mol^{-1}, Molar mass of He = 4 g mol^{-1})

Answer (30)

Sol. Momentum of photon $= \frac{h}{\lambda} = \frac{6.6 \times 10^{-27}}{330 \times 10^{-7}}$ gm cm s^{-1}

Momentum of 1 mole of He-atoms = $m\Delta v$

$$\therefore \quad m\Delta v = N_A \times \frac{h}{\lambda}$$

$$4 \times \Delta v = \frac{6 \times 10^{23} \times 6.6 \times 10^{-27}}{330 \times 10^{-7}}$$

$$\Delta v = \frac{6 \times 6.6 \times 10^2}{33 \times 4} = 30 \text{ cm s}^{-1}$$

$\therefore$ Change in velocity of He-atoms = 30 cm s^{-1}

19. Ozonolysis of ClO_2 produces an oxide of chlorine. The average oxidation state of chlorine in this oxide is ____.

Answer (6)

Sol. ClO_2 contains an odd electron and is paramagnetic. It reacts with ozone to give O_2 and Cl_2O_6.

$2ClO_2 + 2O_3 \rightarrow Cl_2O_6 + 2O_2$

In Cl_2O_6, the average oxidation state of Cl is +6.

PART-III : MATHEMATICS

SECTION - 1 (Maximum Marks : 24)

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONE OR MORE THAN ONE** of these four option(s) is (are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks	: +4	If only (all) the correct option(s) is(are) chosen;
Partial Marks	: +3	If all the four options are correct but ONLY three options are chosen;
Partial Marks	: +2	If three or more options are correct but ONLY two options are chosen, both of which are correct;
Partial Marks	: +1	If two or more options are correct but ONLY one option is chosen and it is a correct option;
Zero Marks	: 0	If unanswered;
Negative Marks	: – 2	In all other cases.

- For example, in a question, if (A), (B) and (D) are the ONLY three options corresponding to correct answers, then
choosing ONLY (A), (B) and (D) will get +4 marks;
choosing ONLY (A) and (B) will get +2 marks;
choosing ONLY (A) and (D) will get +2 marks;
choosing ONLY (B) and (D) will get +2 marks;
choosing ONLY (A) will get +1 mark;
choosing ONLY (B) will get +1 mark;
choosing ONLY (D) will get +1 mark;
choosing no option(s) (i.e. the question is unanswered) will get 0 marks and
choosing any other option(s) will get “2 marks.

1. Let $S_1 = \{(i,j,k) : i,j,k \in \{1,2,...,10\}\}$,

$S_2 = \{(i,j) : 1 \le i < j+2 \le 10, i,j \in \{1,2,...,10\}\}$,

$S_3 = \{(i,j,k,l) : 1 \le i < j < k < l, i,j,k,l \in \{1,2,...,10\}\}$,

and $S_4 = \{(i,j,k,l) :$ i, j, k and l are distinct elements in $\{1,2,...,10\}\}$.

If the total number of elements in the set S_r is n_r, r = 1, 2,3,4, then which of the following statements is (are) TRUE?

(A) $n_1 = 1000$
(B) $n_2 = 44$
(C) $n_3 = 220$
(D) $\frac{n_4}{12} = 420$

Answer (A,B,D)

Sol. Number of elements in $S_1 = 10 \times 10 \times 10 = 1000$

Number of elements in $S_2 = 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 = 44$

Number of elements in $S_3 = {}^{10}C_4 = 210$

Number of elements in $S_4 = {}^{10}P_4 = 210 \times 4! = 5040$

2. Consider a triangle PQR having sides of lengths p, q and r opposite to the angles P, Q and R, respectively. Then which of the following statements is (are) TRUE ?

(A) $\cos P \geq 1 - \dfrac{p^2}{2qr}$

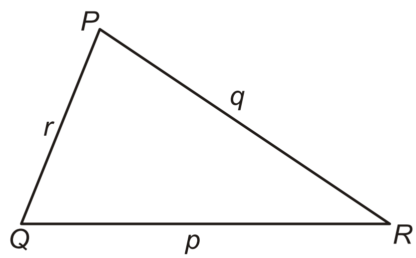
(B) $\cos R \geq \left(\dfrac{q-r}{p+q}\right)\cos P + \left(\dfrac{p-r}{p+q}\right)\cos Q$

(C) $\dfrac{q+r}{p} < 2\dfrac{\sqrt{\sin Q \sin R}}{\sin P}$

(D) If $p < q$ and $p < r$, then $\cos Q > \dfrac{p}{r}$ and $\cos R > \dfrac{p}{q}$

Answer (A,B)

Sol.



$$\cos P = \frac{q^2+r^2-P^2}{2qr} \quad \text{and} \quad \frac{q^2+r^2}{2} \geq \sqrt{q^2.r^2} \quad (\text{AM} \geq \text{GM})$$

$\Rightarrow \quad q^2 + r^2 \geq 2qr$

So, $\cos P \geq \dfrac{2qr - p^2}{2qr}$

$\cos P \geq 1 - \dfrac{p^2}{2qr}$ (A)

(B) $\dfrac{(q-r)\cos P + (p-r)\cos Q}{p+q} = \dfrac{(q\cos P + p\cos Q) - r(\cos P + \cos Q)}{p+q}$

$= \dfrac{r(1-\cos P - \cos Q)}{p+q} = \dfrac{r(q - p\cos R) - (p - q\cos R)}{p+q} = \dfrac{(r-p-q)+(p+q)\cos R}{p+q}$

$= \cos R + \dfrac{r-q-p}{p+q} \leq \cos R \quad (\because r < p+q)$

(C) $\dfrac{q+r}{p} = \dfrac{\sin Q + \sin R}{\sin P} \geq \dfrac{2\sqrt{\sin Q \cdot \sin R}}{\sin P}$

(D) If $p < q$ and $q < r$

So, p is the smallest side, therefore one of Q or R can be obtuse

So, one of $\cos Q$ or $\cos R$ can be negative

Therefore $\cos Q > \dfrac{p}{r}$ and $\cos R > \dfrac{p}{q}$ cannot hold always.

3. Let $f:\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\to\mathbb{R}$ be a continuous function such that $f(0)=1$ and $\int_0^{\frac{\pi}{3}} f(t)dt=0$

Then which of the following statements is (are) TRUE?

(A) The equation $f(x) - 3\cos 3x = 0$ has at least one solution in $\left(0,\frac{\pi}{3}\right)$

(B) The equation $f(x) - 3\sin 3x = -\frac{6}{\pi}$ has at least one solution in $\left(0,\frac{\pi}{3}\right)$

(C) $\lim\limits_{x\to 0}\frac{x\int_0^x f(t)dt}{1-e^{x^2}}=-1$

(D) $\lim\limits_{x\to 0}\frac{\sin x\int_0^x f(t)dt}{x^2}=-1$

Answer (A,B,C)

Sol. $f(0)=1$, $\int_0^{\frac{\pi}{3}} f(t)dt=0$

(A) Consider a function $g(x)=\int_0^x f(t)dt-\sin 3x$

$g(x)$ is continuous and differentiable function

and $g(0) = 0$

$g\left(\frac{\pi}{3}\right)=0$

By Rolle's theorem $g'(x) = 0$ has at least one solution in $\left(0,\frac{\pi}{3}\right)$

$f(x)-3\cos 3x=0$ for some $x\in\left(0,\frac{\pi}{3}\right)$

(B) Consider a function

$h(x)=\int_0^x f(t)dt+\cos 3x+\frac{6}{\pi}x$

$h(x)$ is continuous and differentiable function

and $h(0) = 1$

$h\left(\frac{\pi}{3}\right)=1$

By Rolle's theorem $h'(x) = 0$ for at least one $x\in\left(0,\frac{\pi}{3}\right)$

$f(x)-3\sin 3x+\frac{6}{\pi}=0$ for some $x\in\left(0,\frac{\pi}{3}\right)$

(C) $\lim\limits_{x\to 0}\frac{x\int_0^x f(t)dt}{1-e^{x^2}}$, $\left(\frac{0}{0}\text{form}\right)$

By L' Hopital rule

$$\lim_{x\to 0}\frac{xf(x)+\int_0^x f(t)dt}{-2xe^{x^2}},\left(\frac{0}{0}\text{form}\right)$$

$$=\lim_{x\to 0}\frac{xf'(x)+f(x)+f(x)}{-4x^2e^{x^2}-2e^{x^2}}=\frac{0+2f(0)}{-0-2}=-1$$

(D) $$\lim_{x\to 0}\frac{\sin x\int_0^x f(t)dt}{x^2},\left(\frac{0}{0}\text{form}\right)$$

$$=\lim_{x\to 0}\frac{\sin x\cdot f(x)+\cos x\int_0^x f(t)dt}{2x}$$

$$=\lim_{x\to 0}\frac{\left(\cos x.f(x)+\sin x.f'(x)+\cos x.f(x)-\sin x.\int_0^x f(t)dt\right)}{2}$$

$$=\frac{1+0+1-0}{2}$$

$$=1$$

4. For any real numbers α and β, let $y_{\alpha,\beta}(x)$, $x\in\mathbb{R}$, be the solution of the differential equation $\frac{dy}{dx}+\alpha y=xe^{\beta x}, y(1)=1$

Let $S=\{y_{\alpha,\beta}(x):\alpha,\beta\in\mathbb{R}\}$. Then which of the following functions belong(s) to the set S?

(A) $f(x)=\frac{x^2}{2}e^{-x}+\left(e-\frac{1}{2}\right)e^{-x}$

(B) $f(x)=-\frac{x^2}{2}e^{-x}+\left(e+\frac{1}{2}\right)e^{-x}$

(C) $f(x)=\frac{e^x}{2}\left(x-\frac{1}{2}\right)+\left(e-\frac{e^2}{4}\right)e^{-x}$

(D) $f(x)=\frac{e^x}{2}\left(\frac{1}{2}-x\right)+\left(e+\frac{e^2}{4}\right)e^{-x}$

Answer (A, C)

Sol. $\frac{dy}{dx}+\alpha y=xe^{\beta x}$

Integrating factor (I.F.) = $e^{\int\alpha dx}=e^{\alpha x}$

So, the solution is $y\cdot e^{\alpha x}=\int xe^{\beta x}\cdot e^{\alpha x}dx$

$ye^{\alpha x}=\int xe^{(\alpha+\beta)x}dx$

If $\alpha+\beta\neq 0$

$$ye^{\alpha x} = x\frac{e^{(\alpha+\beta)x}}{(\alpha+\beta)} - \frac{e^{(\alpha+\beta)x}}{(\alpha+\beta)^2} + C$$

$$y = \frac{xe^{\beta x}}{(\alpha+\beta)} - \frac{e^{\beta x}}{(\alpha+\beta)^2} + Ce^{-\alpha x}$$

$$y = \frac{e^{\beta x}}{(\alpha+\beta)}\left(x - \frac{1}{\alpha+\beta}\right) + Ce^{-\alpha x} \quad \text{... (i)}$$

Put $\alpha = \beta = 1$ in (i)

$$y = \frac{e^x}{2}\left(x - \frac{1}{2}\right) + Ce^{-x}$$

$y(1) = 1$

$$1 = \frac{e}{2}\times\frac{1}{2} + \frac{C}{e} \Rightarrow C = e - \frac{e^2}{4}$$

So, $y = \frac{e^x}{2}\left(x - \frac{1}{2}\right) + \left(e - \frac{e^2}{4}\right)e^{-x}$

If $\alpha + \beta = 0$ & $\alpha = 1$

$$\frac{dy}{dx} + y = xe^{-x}$$

I.F. = e^x

$$ye^x = \int xdx$$

$$ye^x = \frac{x^2}{2} + C$$

$$y = \frac{x^2}{2}e^{-x} + Ce^{-x}$$

$y(1) = 1$

$$1 = \frac{1}{2e} + \frac{C}{e} \Rightarrow C = e - \frac{1}{2}$$

$$y = \frac{x^2}{2}e^{-x} + \left(e - \frac{1}{2}\right)e^{-x}$$

5. Let O be the origin and $\overrightarrow{OA} = 2\hat{i} + 2\hat{j} + \hat{k}$, $\overrightarrow{OB} = \hat{i} - 2\hat{j} + 2\hat{k}$ and $\overrightarrow{OC} = \frac{1}{2}(\overrightarrow{OB} - \lambda\overrightarrow{OA})$ for some $\lambda > 0$. If $\left|\overrightarrow{OB}\times\overrightarrow{OC}\right| = \frac{9}{2}$, then which of the following statements is (are) TRUE ?

(A) Projection of $\overrightarrow{OC}$ on $\overrightarrow{OA}$ is $-\frac{3}{2}$

(B) Area of the triangle OAB is $\frac{9}{2}$

(C) Area of the triangle ABC is $\frac{9}{2}$

(D) The acute angle between the diagonals of the parallelogram with adjacent sides $\overrightarrow{OA}$ and $\overrightarrow{OC}$ is $\frac{\pi}{3}$

Answer (A,B,C)

Sol. $\overrightarrow{OA} = 2\hat{i} + 2\hat{j} + \hat{k}$

$\overrightarrow{OB} = \hat{i} - 2\hat{j} + 2\hat{k}$

$\overrightarrow{OC} = \frac{1}{2}(\overrightarrow{OB} - \lambda\overrightarrow{OA})$

$\overrightarrow{OB} \times \overrightarrow{OC} = \overrightarrow{OB} \times \frac{1}{2}(\overrightarrow{OB} - \lambda\overrightarrow{OA}) = -\frac{\lambda}{2}\overrightarrow{OB} \times \overrightarrow{OA} = \frac{\lambda}{2}(\overrightarrow{OA} \times \overrightarrow{OB})$

Now, $\overrightarrow{OA} \times \overrightarrow{OB} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{vmatrix} = 6\hat{i} - 3\hat{j} - 6\hat{k}$

So, $\overrightarrow{OB} \times \overrightarrow{OC} = \frac{3\lambda}{2}(2\hat{i} - \hat{j} - 2\hat{k})$

$\left|\overrightarrow{OB} \times \overrightarrow{OC}\right| = \left|\frac{9\lambda}{2}\right| = \frac{9}{2}$

So, $\lambda = 1 \qquad (\because \lambda > 0)$

$\overrightarrow{OC} = \frac{1}{2}(\overrightarrow{OB} - \overrightarrow{OA})$

$\overrightarrow{OC} = \frac{1}{2}(-\hat{i} - 4\hat{j} + \hat{k})$

(A) Projection of $\overrightarrow{OC}$ on $\overrightarrow{OA}$ = $\frac{\overrightarrow{OC} \cdot \overrightarrow{OA}}{\left|\overrightarrow{OA}\right|} = \frac{\frac{1}{2}(-2-8+1)}{3} = -\frac{3}{2}$

(B) Area of the triangle OAB = $\frac{1}{2}\left|\overrightarrow{OA} \times \overrightarrow{OB}\right| = \frac{9}{2}$

(C) Area of the triangle ABC is = $\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\left\|\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -4 & 1 \\ -\frac{5}{2} & -4 & -\frac{1}{2} \end{vmatrix}\right\| = \frac{1}{2}\left|6\hat{i} - 3\hat{j} - 6\hat{k}\right| = \frac{9}{2}$

(D) Acute angle between the diagonals of the parallelogram with adjacent sides $\overrightarrow{OA}$ and $\overrightarrow{OC}$ $= \theta$

$\frac{(\overrightarrow{OA} + \overrightarrow{OC}) \cdot (\overrightarrow{OA} - \overrightarrow{OC})}{\left|\overrightarrow{OA} + \overrightarrow{OC}\right| \left|\overrightarrow{OA} - \overrightarrow{OC}\right|} = \cos\theta$

$\cos\theta = \frac{\left(\frac{3}{2}\hat{i} + \frac{3}{2}\hat{k}\right) \cdot \left(\frac{5}{2}\hat{i} + 4\hat{j} + \frac{1}{2}\hat{k}\right)}{\frac{3}{2}\sqrt{2} \times \sqrt{\frac{90}{4}}} = \frac{18}{3\sqrt{2}\sqrt{90}}$

$\theta \neq \frac{\pi}{3}$

6. Let E denote the parabola $y^2 = 8x$. Let $P = (-2, 4)$, and let Q and Q' be two distinct points on E such that the lines PQ and PQ' are tangents to E. Let F be the focus of E. Then which of the following statements is (are) TRUE ?

(A) The triangle PFQ is a right-angled triangle

(B) The triangle QPQ' is a right-angled triangle

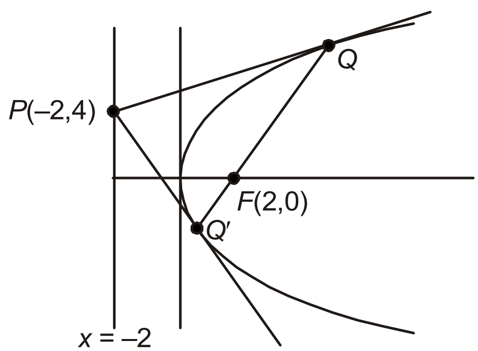
(C) The distance between P and F is $5\sqrt{2}$

(D) F lies on the line joining Q and Q'

Answer (A,B,D)

Sol. $E : y^2 = 8x$

$P : (-2, 4)$



Point P $(-2, 4)$ lies on directrix $(x = -2)$ of parabola $y^2 = 8x$

So, $\angle QPQ' = \frac{\pi}{2}$ and chord QQ' is a focal chord and segment PQ subtends right angle at the focus.

Slope of $QQ' = \frac{2}{t_1 + t_2} = 1$

Slope of $PF = -1$

$PF = 4\sqrt{2}$

SECTION - 2 (Maximum Marks : 12)

- This section contains **THREE (03)** question stems.
- There are **TWO (02)** questions corresponding to each question stem.
- The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical value corresponding to the answer in the designated place using the mouse and the on-screen virtual numeric keypad.
- If the numerical value has more than two decimal places, **truncate/round-off** the value to **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks	:	+2	If ONLY the correct numerical value is entered.
Zero Marks	:	0	In all other cases.

Question Stem for Question Nos. 7 and 8

Consider the region $R = \{(x, y) \in \mathrm{R} \times \mathrm{R} : x \geq 0 \text{ and } y^2 \leq 4 - x\}$. Let F be the family of all circles that are contained in R and have centers on the x-axis. Let C be the circle that has largest radius among the circles in F. Let (α, β) be a point where the circle C meets the curve $y^2 = 4 - x$.

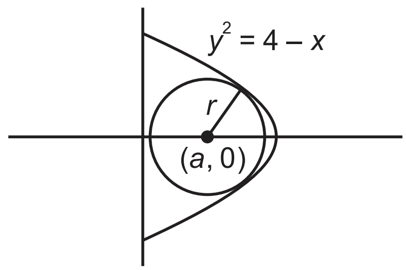
7. The radius of the circle C is _____.

Answer (1.50)

8. The value of α is _____.

Answer (2.00)

Sol. For comprehension Q7 & Q8



Let the circle be,

$(x - a)^2 + y^2 = r^2$

Solving it with parabola

$y^2 = 4 - x$ we get

$(x - a)^2 + 4 - x = r^2$

$\Rightarrow$ $x^2 - x(2a + 1) + (a^2 + 4 - r^2) = 0$ ...(1)

$D = 0$

$\Rightarrow$ $4r^2 + 4a - 15 = 0$

Clearly $a \geq r$

So $4r^2 + 4r - 15 \leq 0$

$\Rightarrow$ $r_{\max} = \frac{3}{2} = a$

Radius of circle C is $\frac{3}{2}$

From (1) $x^2 - 4x + 4 = 0$

$\Rightarrow$ $x = 2 = \alpha$

Question Stem for Question Nos. 9 and 10

Let $f_1 : (0, \infty) \to \mathrm{R}$ and $f_2 : (0, \infty) \to R$ be defined by $f_1(x) = \int_0^x \prod_{j=1}^{21} (t - j)^j \, dt, \, x > 0$

and $f_2(x) = 98(x - 1)^{50} - 600(x - 1)^{49} + 2450$, $x > 0$, where, for any positive integer n and real number $a_1, a_2, \ldots a_n$, $\prod_{i=1}^{n} a_i$ denotes the product of $a_1, a_2, \ldots a_n$. Let m_i and n_i, respectively, denote the number of points of local minima and the number of points of local maxima of function f_i, $i = 1, 2$, in the interval $(0, \infty)$.

Solution for Q9 and 10

$$f_1'(x) = \prod_{j=1}^{21}(x-j)^j$$

$$f_1'(x) = (x-1)(x-2)^2(x-3)^3,...,(x-20)^{20}(x-21)^{21}$$

Checking the sign scheme of $f_1'(x)$ at x = 1, 2, 3, ..., 21, we get

$f_1(x)$ has local minima at x = 1, 5, 9, 13, 17, 21 and local maxima at x = 3, 7, 11, 15, 19

$\Rightarrow$ $m_1 = 6$, $n_1 = 5$

$f_2(x) = 98(x-1)^{50} - 600(x-1)^{49} + 2450$

$$f_2'(x) = 98 \times 50(x-1)^{49} - 600 \times 49 \times (x-1)^{48}$$

$= 98 \times 50 \times (x-1)^{48}(x-7)$

$f_2(x)$ has local minimum at $x = 7$ and no local maximum.

$\Rightarrow$ $m_2 = 1$, $n_2 = 0$

9. The value of $2m_1 + 3n_1 + m_1n_1$ is _____.

Answer (57.00)

Sol. $2m_1 + 3n_1 + m_1n_1$

$= 2 \times 6 + 3 \times 5 + 6 \times 5$

$= 57$

10. The value of $6m_2 + 4n_2 + 8m_2n_2$ is _____.

Answer (06.00)

Sol. $6m_2 + 4n_2 + 8m_2n_2$

$= 6 \times 1 + 4 \times 0 + 8 \times 1 \times 0 = 6$

Question Stem for Question Nos. 11 and 12

Let $g_i : \left[\frac{\pi}{8}, \frac{3\pi}{8}\right] \to \mathbb{R}, i = 1, 2,$ and $f : \left[\frac{\pi}{8}, \frac{3\pi}{8}\right] \to \mathbb{R}$ be functions such that $g_1(x) = 1$, $g_2(x) = |4x - \pi|$ and $f(x) = \sin^2 x$, for all $x \in \left[\frac{\pi}{8}, \frac{3\pi}{8}\right]$. Define $S_i = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} f(x) \cdot g_i(x)\,dx, i = 1, 2$

11. The value of $\frac{16S_1}{\pi}$ is _____.

Answer (02.00)

Sol. $S_1 = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \sin^2 x . 1 dx$

$$= \frac{1}{2}\int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} (1 - \cos 2x)\,dx$$

$$=\frac{1}{2}\left(x-\frac{\sin 2x}{2}\right)_{\frac{\pi}{8}}^{\frac{3\pi}{8}}$$

$$S_1=\frac{1}{2}\left(\frac{\pi}{4}-0\right)=\frac{\pi}{8}$$

$$\Rightarrow \frac{16S_1}{\pi}=2$$

12. The value of $\frac{48S_2}{\pi^2}$ is _____.

Answer (01.50)

Sol. $S_2=\int_{\frac{\pi}{8}}^{\frac{3\pi}{8}}\sin^2 x.|4x-\pi|dx$

$$=\int_{\frac{\pi}{8}}^{\frac{3\pi}{8}}4\sin^2 x\left|x-\frac{\pi}{4}\right|dx$$

Let $x-\frac{\pi}{4}=t \Rightarrow dx=dt$

$$S_2=\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}}4\sin^2\left(\frac{\pi}{4}+t\right)|t|dt$$

$$=\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}}2\left(1-\cos 2\left(\frac{\pi}{4}+t\right)\right)|t|dt$$

$$=\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}}(2+2\sin 2t)|t|dt$$

$$=2\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}}|t|dt+2\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}}|t|\sin(2t)dt$$

$$=4\int_{0}^{\frac{\pi}{8}}tdt+0$$

$$S_2=2t^2\Big]_0^{\frac{\pi}{8}}=\frac{\pi^2}{32}$$

$$\Rightarrow \frac{48S_2}{\pi^2}=\frac{3}{2}$$

SECTION - 3 (Maximum Marks : 12)

- This section contains **TWO (02) paragraphs**. Based on each paragraph, there are **TWO (02) questions.**
- Each question has **FOUR** options (A), (B), (C) and (D). **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks	:	+3	If ONLY the correct option is chosen;
Zero Marks	:	0	If none of the options is chosen (i.e. the question is unanswered);
Negative Marks	:	-1	In all other cases.

Paragraph

Let $M = \{(x, y) \in R \times R : x^2 + y^2 \le r^2\}$, where $r > 0$. Consider the geometric progression $a_n = \frac{1}{2^{n-1}}$, n = 1, 2, 3, ... Let $S_0 = 0$ and, for $n \ge 1$, let S_n denote the sum of the first n terms of this progression. For $n \ge 1$, let C_n denote the circle with center $(S_{n-1}, 0)$ and radius a_n, and D_n denote the circle with center (S_{n-1}, S_{n-1}) and radius a_n.

13. Consider M with $r = \frac{1025}{513}$. Let k be the number of all those circles C_n that are inside M. Let l be the maximum possible number of circles among these k circles such that no two circles intersect. Then

(A) $k + 2l = 22$

(B) $2k + l = 26$

(C) $2k + 3l = 34$

(D) $3k + 2l = 40$

Answer (D)

Sol. $\because \quad a_n = \frac{1}{2^{n-1}}$ and $S_n = 2\left(1 - \frac{1}{2^n}\right)$

For circles C_n to be inside M.

$$S_{n-1} + a_n < \frac{1025}{513}$$

$$\Rightarrow \quad S_n < \frac{1025}{513}$$

$$\Rightarrow \quad 1 - \frac{1}{2^n} < \frac{1025}{1026} = 1 - \frac{1}{1026}$$

$\Rightarrow \quad 2^n < 1026$

$\Rightarrow \quad n \le 10$

$\therefore \quad$ Number of circles inside be 10 = K

Clearly alternate circles do not intersect each other *i.e.*, C_1, C_3, C_5, C_7, C_9 do not intersect each other as well as C_2, C_4, C_6, C_8 and C_{10} do not intersect each other hence maximum 5 set of circles do not intersect each other.

$\therefore \quad l = 5$

$\therefore \quad 3K + 2l = 40$

$\therefore \quad$ Option (D) is correct

14. Consider M with $r = \frac{(2^{199}-1)\sqrt{2}}{2^{198}}$. The number of all those circles D_n that are inside M is

(A) 198 (B) 199

(C) 200 (D) 201

Answer (B)

Sol. $\because \quad r = \frac{(2^{199}-1)\sqrt{2}}{2^{198}}$

Now, $\sqrt{2}\, S_{n-1} + a_n < \left(\frac{2^{199}-1}{2^{198}}\right)\sqrt{2}$

$$2\cdot\sqrt{2}\left(1-\frac{1}{2^{n-1}}\right)+\frac{1}{2^{n-1}}<\left(\frac{2^{199}-1}{2^{198}}\right).$$

$$\therefore \quad 2\sqrt{2}-\frac{\sqrt{2}}{2^{n-2}}+\frac{1}{2^{n-1}}<2\sqrt{2}-\frac{\sqrt{2}}{2^{198}}$$

$$\frac{1}{2^{n-2}}\left(\frac{1}{2}-\sqrt{2}\right)<-\frac{\sqrt{2}}{2^{198}}$$

$$\frac{2\sqrt{2}-1}{2\cdot 2^{n-2}}>\frac{\sqrt{2}}{2^{198}}$$

$$2^{n-2}<\left(2-\frac{1}{\sqrt{2}}\right)2^{197}$$

$\therefore \quad n \le 199$

$\therefore \quad$ Number of circles = 199

Option (B) is correct.

Paragraph

Let $\psi_1 : [0, \infty) \to \mathbb{R}$, $\psi_2 : [0, \infty) \to \mathbb{R}$, $f : [0, \infty) \to \mathbb{R}$ and $g : [0, \infty) \to \mathbb{R}$ be functions such that $f(0) = g(0) = 0$,

$$\psi_1(x) = e^{-x} + x,\ x \ge 0,$$

$$\psi_2(x) = x^2 - 2x - 2e^{-x} + 2,\ x \ge 0,$$

$$f(x) = \int_{-x}^{x}\left(|t| - t^2\right)e^{-t^2}\,dt,\ x > 0$$

and $\quad g(x) = \int_0^{x^2}\sqrt{t}\, e^{-t}\, dt,\ x > 0$.

15. Which of the following statements is TRUE?

(A) $f\left(\sqrt{\ln 3}\right) + g\left(\sqrt{\ln 3}\right) = \frac{1}{3}$

(B) For every $x > 1$, there exists an $\alpha \in (1, x)$ such that $\Psi_1(x) = 1 + \alpha x$

(C) For every $x > 0$, there exists a $\beta \in (0, x)$ such that $\Psi_2(x) = 2x\,(\Psi_1(\beta) - 1)$

(D) f is an increasing function on the interval $\left[0, \frac{3}{2}\right]$

Answer (C)

Sol. $\because \quad g(x)=\int_0^{x^2}\sqrt{t}\,e^{-t}\,dt,\ \ x>0$

Let $t = u^2 \Rightarrow dt = 2u\,du$

$$\therefore \quad g(x)=\int_0^{x} u\,e^{-u^2}\cdot 2u\,du$$

$$=2\int_0^{x} t^2\,e^{-t^2}dt \qquad \text{...(i)}$$

and $f(x)=\int_{-x}^{x}\left(|t|-t^2\right)e^{-t^2}dt,\ x>0$

$$\therefore \quad f(x)=2\int_0^{x}\left(t-t^2\right)e^{-t^2}dt \qquad \text{...(ii)}$$

From equation (i) + (ii) : $f(x)+g(x)=\int_0^{x}2te^{-t^2}\,dt$

Let $t^2 = P \quad \Rightarrow \quad 2t\,dt = dP$

$$\therefore \quad f(x)+g(x)=\int_0^{x^2}e^{-P}dP=\left[-e^{-P}\right]_0^{x^2}$$

$$\therefore \quad f(x)+g(x)=1-e^{-x^2} \qquad \text{...(iii)}$$

$$\therefore \quad f\left(\sqrt{\ln 3}\right)+g\left(\sqrt{\ln 3}\right)=1-e^{-\ln 3}=1-\frac{1}{3}=\frac{2}{3}$$

$\therefore$ Option (A) is incorrect.

From equation (ii) : $f'(x)=2\left(x-x^2\right)e^{-x^2}=2x(1-x)e^{-x^2}$

$\because$ $f(x)$ is increasing in (0, 1)

$\therefore$ Option (D) is incorrect

$\because$ $\Psi_1(x) = e^{-x} + x$

$\Rightarrow$ $\Psi'_1(x) = 1 - e^{-x} < 1$ for $x > 1$

Then for $\alpha \in (1,\ x)$, $\Psi_1(x)=1+\alpha x$ does not true for $\alpha > 1$.

$\therefore$ Option (B) is incorrect

Now $\psi_2(x)=x^2-2x-2e^{-x}+2$

$$\Rightarrow \quad \psi_2'(x)=2x-2+2e^{-x}$$

$$\therefore \quad \psi_2'(x)=2\psi_1(x)-2$$

From LMVT

$$\frac{\psi_2(x)-\psi_2(0)}{x-0}=\psi_2'(\beta) \text{ for } \beta\in(\infty,\ x)$$

$$\Rightarrow \quad \psi_2(x)=2x(\psi_1(\beta)-1)$$

$\therefore$ Option (C) is correct.

16. Which of the following statements is TRUE?

(A) $\Psi_1(x) \le 1$, for all $x > 0$

(B) $\Psi_2(x) \le 0$, for all $x > 0$

(C) $f(x) \ge 1 - e^{-x^2} - \frac{2}{3}x^3 + \frac{2}{5}x^5$, for all $x \in \left(0, \frac{1}{2}\right)$

(D) $g(x) \le \frac{2}{3}x^3 - \frac{2}{5}x^5 + \frac{1}{7}x^7$, for all $x \in \left(0, \frac{1}{2}\right)$

Answer (D)

Sol. $\because \quad \Psi_1(x) = e^{-x} + x$

and for all x > 0, $\Psi_1(x) > 1$

$\therefore$ (A) is not correct

$\Psi_1(x) = x^2 + 2 - 2\,(e^{-x} + x) > 0$ for $x > 0$

$\therefore$ (B) is not correct

Now, $\sqrt{t}\, e^{-t} = \sqrt{t}\left(1 - \frac{t}{1!} + \frac{t^2}{2!} - \frac{t^3}{3!} + ...\infty\right)$

and $\sqrt{t}\, e^{-t} \le t^{\frac{1}{2}} - t^{\frac{3}{2}} + \frac{1}{2}t^{\frac{5}{2}}$

$$\therefore \quad \int_0^{x^2} \sqrt{t}\, e^{-t}\, dt \le \int_0^{x^2} \left(t^{\frac{1}{2}} - t^{\frac{3}{2}} + \frac{1}{2}t^{\frac{5}{2}}\right) dt$$

$$= \frac{2}{3}x^3 - \frac{2}{3}x^5 + \frac{1}{7} + \frac{1}{7}x^7$$

$\therefore$ Option (D) is correct

and $f(x) = \int_{-x}^{x} (|t| - t^2)e^{-t^2} dt$

$$= 2\int_0^x (t - t^2)e^{-t^2} dt$$

$$= \int_0^x 2te^{-t^2} dt - 2\int_0^x t^2 e^{-t^2} dt$$

$$= 1 - e^{-x^2} - 2\int_0^x t^2 e^{-t^2} dt$$

$$\therefore \quad f(x) \le 1 - e^{-x^2} - 2\int_0^x t^2\left(1 - t^2\right) dt$$

$$= 1 - e^{-x^2} - 2\frac{x^3}{3} + \frac{2}{5}x^5 \text{ for all } x\left(0, \frac{1}{2}\right)$$

$\therefore$ Option (C) is incorrect.

SECTION - 4 (Maximum Marks : 12)

- This section contains **THREE (03)** questions.
- The answer to each question is a **NON-NEGATIVE INTEGER.**
- For each question, enter the correct integer corresponding to the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks	:	+4	If ONLY the correct integer is entered;
Zero Marks	:	0	In all other cases.

17. A number is chosen at random from the set {1, 2, 3......, 2000}. Let p be the probability that the number is a multiple of 3 or a multiple of 7. Then the value of $500p$ is ______.

Answer (214)

Sol. E = a number which is multiple of 3 or multiple of 7

$n(E)$ = (3, 6, 9, ..., 1998) + (7, 14, 21, ..., 1995) – (21, 42, 63, ... 1995)

$n(E)$ = 666+ 285 – 95

$n(E)$ = 856

$n(E)$ = 2000

$$P(E) = \frac{856}{2000}$$

$$P(E) \times 500 = \frac{856}{4} = 214$$

18. Let E be the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$. For any three distinct points P, Q and Q' on E, let M (P, Q) be the mid-point of the line segment joining P and Q, and $M(P, Q')$ be the mid-point of the line segment joining P and Q'. Then the maximum possible value of the distance between $M(P, Q)$ and $M(P, Q')$, as P, Q and Q' vary on E, is _____.

Answer (4)

Sol. Let $P(\alpha)$, $Q(\theta)$, $Q'(\theta')$

$$M = \frac{1}{2}(4\cos\alpha + 4\cos\theta),\ \frac{1}{2}(3\sin\alpha + 3\sin\theta)$$

$$M' = \frac{1}{2}(4\cos\alpha + 4\cos\theta'),\ \frac{1}{2}(3\sin\alpha + 3\sin\theta')$$

$$MM' = \frac{1}{2}\sqrt{(4\cos\theta - 4\cos\theta')^2 + (3\sin\theta - 3\sin\theta')^2}$$

$$MM' = \frac{1}{2}\text{ distance between } Q \text{ and } Q'$$

MM' is not depending on P

Maximum of QQ' is possible when QQ' = major axis

QQ' = 2(4) = 8

$$MM' = \frac{1}{2}\cdot(QQ')$$

MM' = 4

19. For any real number x, let $[x]$ denote the largest integer less than or equal to x. If $I = \int_0^{10} \left[\sqrt{\frac{10x}{x+1}}\right] dx$, then the value of $9I$ is ________.

Answer (182.00)

Sol. $I = \int_0^{10} \left[\sqrt{\frac{10x}{x+1}}\right] dx$

$$y = \frac{10x}{x+1} \quad , \quad 0 \le x \le 10$$

$$xy + y = 10x$$

$$x = \frac{y}{10-y}$$

$$0 \le \frac{y}{10-y} \le 10$$

$$\frac{y}{10-y} \ge 0 \quad \text{and} \quad \frac{y}{10-y} - 10 \le 0$$

$$\frac{y}{y-10} \le 0 \quad \text{and} \quad \frac{11y-100}{y-10} \ge 0$$

$+ \quad - \quad +$; $0 \quad 10$ and $+ \quad - \quad +$; $\frac{100}{11} \quad 10$

$$y \in [0, 10) \quad \text{and} \quad y \in \left(-\infty, \frac{100}{11}\right] \cup (10, \infty)$$

$$y \in \left[0, \frac{100}{11}\right]$$

$$\sqrt{y} \in \left[0, \frac{10}{\sqrt{11}}\right] \quad \Rightarrow \quad \left[\sqrt{y}\right] = \{0, 1, 2, 3\}$$

Cose I : $0 \le \frac{10x}{x+1} < 1$

$$\frac{10x}{x+1} \ge 0 \quad \text{and} \quad \frac{10x}{x+1} - 1 < 0$$

$+ \quad - \quad +$; $-1 \quad 0$ and $\frac{9x-1}{x+1} < 0$

$+ \quad - \quad +$; $-1 \quad \frac{1}{9}$

$$x \in (-\infty, -1) \cup [0, \infty) \quad \text{and} \quad x \in \left(-1, \frac{1}{9}\right)$$

$$x \in \left[0, \frac{1}{9}\right) \quad \text{then} \quad \left[\sqrt{\frac{10x}{x+1}}\right] = 0$$

Case II : $1 \le \dfrac{10x}{x+1} < 4$

$$\frac{10x}{x+1} - 1 \ge 0 \quad \text{and} \quad \frac{10x}{x+1} - 4 < 0$$

$$\frac{9x-1}{x+1} \ge 0 \quad \text{and} \quad \frac{6x-4}{x+1} < 0$$

$$+ \;\; -1 \;\; - \;\; \frac{1}{9} \;\; + \quad \text{and} \quad + \;\; -1 \;\; - \;\; +\frac{2}{3} \;\; +$$

$$x \in (-\infty, -1) \cup \left[\frac{1}{9}, \infty\right) \quad \text{and} \quad x \in \left(-1, \frac{2}{3}\right)$$

$$x \in \left[\frac{1}{9}, \frac{2}{3}\right) \quad , \quad \left[\sqrt{\frac{10x}{x+1}}\right] = 1$$

Case III : $4 \le \dfrac{10x}{x+1} < 9$

$$\frac{10x}{x+1} - 4 \ge 0 \quad \text{and} \quad \frac{10x}{x+1} < 9$$

$$\frac{6x-4}{x+1} \ge 0 \quad \text{and} \quad \frac{x-9}{x+1} < 0$$

$$+ \;\; -1 \;\; - \;\; \frac{2}{3} \;\; + \quad \text{and} \quad + \;\; -1 \;\; - \;\; 9 \;\; +$$

$$x \in (-\infty, -1) \cup \left[\frac{2}{3}, \infty\right) \qquad x \in (-1, 9)$$

$$x \in \left[\frac{2}{3}, 9\right) \quad ; \quad \left[\sqrt{\frac{10x}{x+1}}\right] = 2$$

Case IV : $x \in [9, 10] \quad \Rightarrow \quad \left[\sqrt{\dfrac{10x}{x+1}}\right] = 3$

$$I = \int_0^{\frac{1}{9}} 0 \cdot dx + \int_{\frac{1}{9}}^{\frac{2}{3}} 1 \cdot dx + \int_{\frac{2}{3}}^{9} 2 \cdot dx + \int_9^{10} 3 \cdot dx$$

$$I = \left(\frac{2}{3} - \frac{1}{9}\right) + 2\left(9 - \frac{2}{3}\right) + 3(10-9)$$

$$I = \frac{5}{9} + \frac{50}{3} + 3$$

$9I = 182$